THE EFFECT OF THE EARTH'S ROTATION ON THE LINEAR GRAVITY WAVE DRAG OF A SHEARED FLOW PAST A CIRCULAR MOUNTAIN

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Abstract: The linear model of mountain wave drag for flow over circular mountains of Teixeira et al. (2004) is extended to take into account the effect of the Earth's rotation. The model is inviscid and hydrostatic and uses the WKB approximation to treat generic wind profiles. The drag normalized by its value in the absence of shear and rotation is calculated here as a function of the Richardson number Ri and of the Rossby number Ro, for a linear wind profile and a wind that turns with height. For the linear wind profile, it is seen that the drag tends to decrease with Ri for high Ro, but tends to increase as Ri decreases when Ro is lower. The peak of the drag as a function of Ro is displaced from $Ro = \infty$ for $Ri = \infty$ to finite values of Ro for lower Ri. The same happens for a wind that turns with height, with the only difference that in this case the drag always increases as Ri decreases.

Keywords: *Linear theory, Mountain wave drag, Rotation, WKB approximation*

1. INTRODUCTION

The effect of wind shear on gravity wave drag (GWD) has been the object of recent linear theoretical studies. One of them, developed by Teixeira and Miranda (2006), provides closed-form analytical GWD formulae for hydrostatic non-rotating flow over elliptical mountains, extending previous results for flow over circular and 2D mountains (Teixeira et al., 2004; Teixeira and Miranda, 2004). These formulae are easily implemented in GWD parametrization schemes, thereby allowing an inclusion of wind profile effects in these schemes. For mountains with widths of 10 km to 100 km, which are still inadequately resolved in many global models, the effects of the Earth's rotation on the GWD are likely to become important. The present study extends the model of Teixeira et al. (2004) for GWD in flow past a circular mountain to a case with rotation. In this situation, there is no general analytical solution to the wave equations, and even the solution for the simple case of a linear wind profile is expressed in terms of hypergeometric functions. This difficulty is eliminated here by assuming that the wind velocity varies sufficiently slowly with height, so that a WKB approximation may be used. Since the effects of rotation are scale-dependent, the GWD now depends on the shape of the orography, and the drag expressions calculated are not analytical, but must be evaluated numerically.

2. THEORETICAL MODEL

An inviscid, hydrostatic atmosphere flowing steadily over a circular bell-shaped mountain is addressed. The Brunt-Väisälä frequency N is assumed to be constant, for simplicity, but generic, slowly varying wind profiles are considered. When the Earth's rotation is taken into account, the equivalent to the Taylor-Goldstein equation for the Fourier transform of the vertical velocity perturbation can be written (cf. Jones, 1967):

$$\left[1-\frac{f^2}{(Uk+Vl)^2}\right]\hat{w}''+\frac{2f^2}{(Uk+Vl)^2}\left(\frac{U'k+V'l}{Uk+Vl}-i\frac{U'l-V'k}{f}\right)\hat{w}'+\left\{\frac{N^2(k^2+l^2)}{(Uk+Vl)^2}-\frac{U''k+V''l}{Uk+Vl}\right.$$
$$\left.+i\frac{f}{Uk+Vl}\left[2\frac{(U'l-V'k)(U'k+V'l)}{(Uk+Vl)^2}-\frac{U''l-V''k}{Uk+Vl}\right]\right\}\hat{w}=0. \quad (1)$$

Here (U,V) is the incoming wind, (k,l) is the horizontal wavenumber of the internal waves, f is the Coriolis parameter and the primes denote differentiation with respect to height, z. The boundary conditions to this equation state that the wave energy radiates upwards at the top of the domain and that the flow follows the terrain elevation at the surface:

$$\hat{w}(z=0)=i(U_0k+V_0l)\hat{h}, \quad (2)$$

where (U_0,V_0) is the surface incoming wind velocity and $\hat{h}$ is the Fourier transform of the terrain elevation.

The solution to (1) is determined in the form

$$\hat{w} = \hat{w}(z=0)\exp\left\{ i\int_0^z [m_0(\zeta) + \varepsilon m_1(\zeta) + \varepsilon^2 m_2(\zeta)]d\zeta \right\}, \tag{3}$$

where the vertical wavenumber of the internal waves has been expanded in a power series of the small parameter ε. Inserting (3) into (1), the zeroth, first and second-order parts of the vertical wavenumber are found to be:

$$m_0 = \frac{N(k^2+l^2)^{1/2}}{[(Uk+Vl)^2 - f^2]^{1/2}}, \tag{4}$$

$$\varepsilon m_1 = i\frac{[2f^2 - (Uk+Vl)^2](U'k+V'l) - 2if(Uk+Vl)(U'l-V'k)}{2(Uk+Vl)[(Uk+Vl)^2 - f^2]}, \tag{5}$$

$$\varepsilon^2 m_2 = \frac{\left[10f^2(Uk+Vl)^2 - (Uk+Vl)^4 - 8f^4\right](U'k+V'l)^2 + 4f^2(Uk+Vl)^2(U'l-V'k)^2}{8N(k^2+l^2)^{1/2}(Uk+Vl)^2\left[(Uk+Vl)^2-f^2\right]^{3/2}} + \frac{\left[4f^4(Uk+Vl) - 2(Uk+Vl)^5 - 2f^2(Uk+Vl)^3\right](U''k+V''l)}{8N(k^2+l^2)^{1/2}(Uk+Vl)^2\left[(Uk+Vl)^2-f^2\right]^{3/2}}. \tag{6}$$

We are interested here in the surface drag, which in general is given by:

$$(D_x, D_y) = 4\pi^2 i \int\int (k,l)\hat{p}^*(z=0)\hat{h}\, dkdl, \tag{7}$$

where $\hat{p}$ is the Fourier transform of the pressure perturbation and the asterisk denotes complex conjugate. The pressure may be obtained from

$$\hat{p} = i\frac{\rho_0}{k^2+l^2}\left\{ \left[\frac{f^2}{Uk+Vl} - (Uk+Vl)\right]\hat{w}' + \left[(U'k+V'l) + if\frac{U'k+V'l}{Uk+Vl}\right]\hat{w} \right\}, \tag{8}$$

whereas various forms for $\hat{h}$ may be chosen, depending on the preferred orography shape. Here a bell-shaped circular mountain will be used, which means that

$$\hat{h} = \frac{h_0 a^2}{2\pi} \mathrm{e}^{-a(k^2+l^2)^{1/2}}, \tag{9}$$

where h_0 is the maximum height of the mountain and a is its half-width.

As the previous expressions would suggest, the explicit form of the drag is in general quite complicated, so two simple flows (already addressed in Teixeira et al., 2004) will be treated: a linearly decreasing unidirectional wind and a wind that turns with height at a constant rate maintaining its magnitude.

3. RESULTS

The linear wind profile considered here is defined as

$$U = U_0 - \alpha z, \quad V = 0, \tag{10}$$

where $U_0 > 0$ and $\alpha > 0$ are constant coefficients. The corresponding Richardson number, which is constant, is $Ri = N^2/\alpha^2$. The wind that turns with height, on the other hand, is defined as

$$U = U_0\cos(\beta z), \quad V = U_0\sin(\beta z), \tag{11}$$

where $\beta > 0$ is a constant coefficient (the wind turns anti-clockwise as z increases). The corresponding Richardson number, which is also constant, is $Ri = N^2/(U_0^2\beta^2)$.

The drag in each of these cases is given by (7), using (8) and the solution (3), together with (2), (4)-(6) and (9). For an easier comparison with previous results, the drag is here normalized by its value without rotation and for a constant wind equal to U_0.

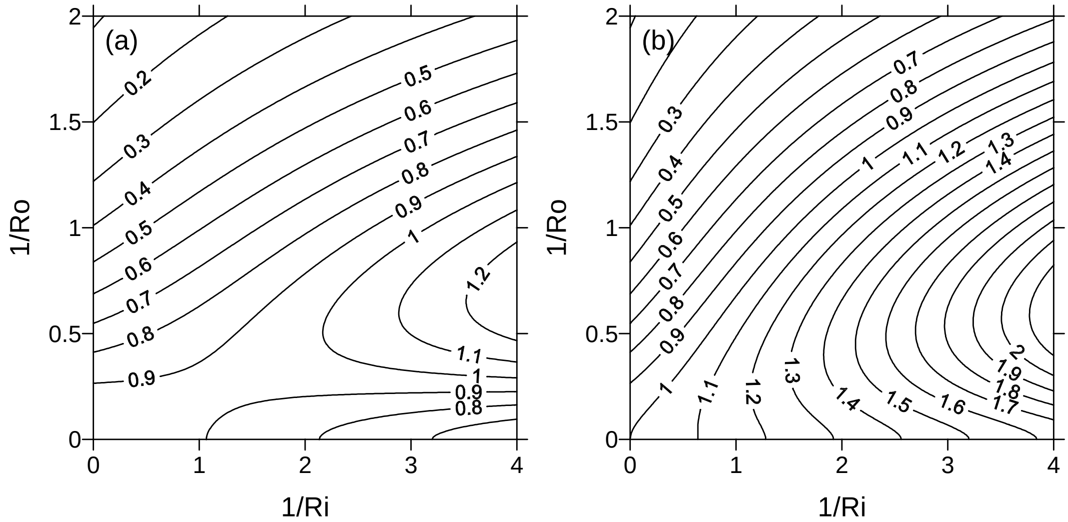

Figure 1: Variation of the drag normalized by its value in the absence of shear and rotation with Ri^{-1} and Ro^{-1}. (a) Linear wind profile. (b) Wind that turns with height.

Figure 1 shows the variation of the drag with Ri^{-1} and $Ro^{-1} = fa/U_0$ for the two previous flows (in Fig. 1(a) and Fig. 1(b), respectively). In Fig. 1(a) it can be seen that the drag generally decreases as Ri^{-1} increases for a linear wind profile when Ro^{-1} is low, and decreases as Ro^{-1} increases when Ri^{-1} is low. But when Ro^{-1} is higher, the drag increases with Ri^{-1}. This dependency changes sign at about $Ro^{-1} = 0.25$. Figure 1(b) shows that for a turning wind the drag increases with Ri^{-1} for every Ro^{-1} displayed, but the dependency on Ri^{-1} becomes considerably stronger, especially in relative terms, for higher values of Ro^{-1}.

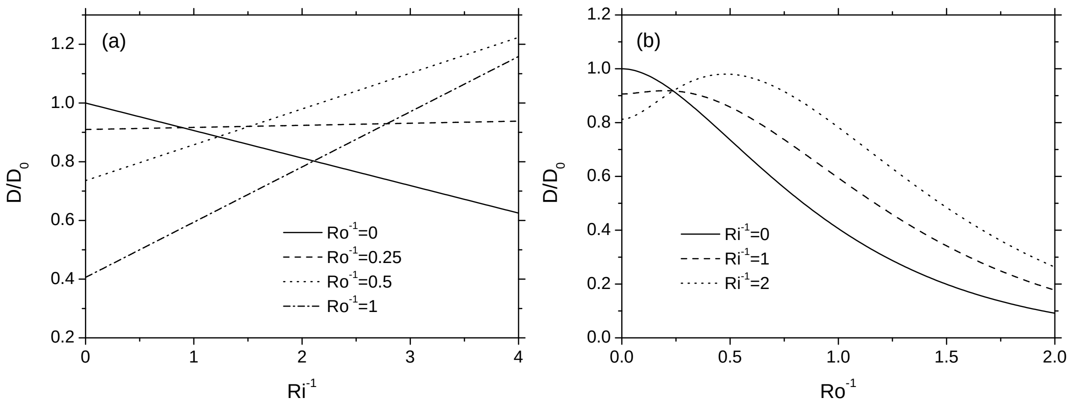

Figure 2: Normalized drag for a linear wind profile. (a) Drag variation with Ri^{-1}, for various Ro^{-1} (see legend). (b) Drag variation with Ro^{-1} for various Ri^{-1} (see legend).

Figure 2 shows more detailed results for the linear wind profile. The variation of the drag with Ri^{-1} for various values of Ro^{-1} is displayed in Fig. 2(a), while Fig. 2(b) shows the variation of the drag with Ro^{-1} for various values of Ri^{-1}. From Fig. 2(a), it can be seen what was already noted in Fig. 1(a): the variation of the drag with Ri^{-1} changes sign from decreasing as Ri^{-1} increases to increasing with Ri^{-1}, as Ro^{-1} becomes higher. Figure 2(b), on the other hand, shows that although for sufficiently large Ro^{-1} the drag eventually decreases with Ro^{-1} the value of Ro^{-1} for which the drag is a maximum is translated from $Ro^{-1} = 0$ to a higher Ro^{-1} as Ri^{-1} increases.

Finally, Fig. 3 shows the same kind to results for a wind that turns with height. In Fig. 3(a), where the variation of the drag with Ri^{-1} for various values of Ro^{-1} is displayed, it can be seen that the drag always increases with Ri^{-1}, but more strongly for higher values of Ro^{-1} (cf. Fig. 1(b)). On the other hand, the

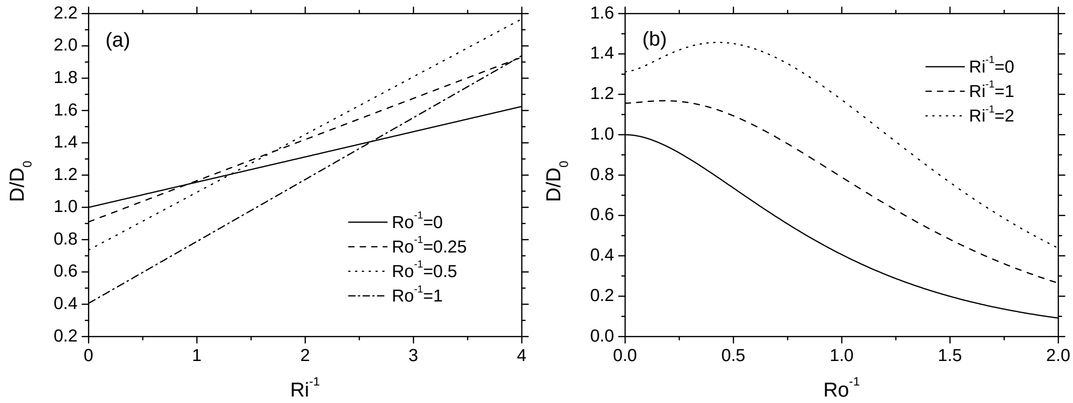

Figure 3: Normalized drag for a wind that turns with height. (a) Drag variation with Ri^{-1}, for various Ro^{-1} (see legend). (b) Drag variation with Ro^{-1} for various Ri^{-1} (see legend).

variation of the drag with Ro^{-1} for various values of Ri^{-1}, displayed in Fig. 3(b), shows a similar behaviour to Fig. 2(b). Namely, the maximum drag value is translated from $Ro^{-1} = 0$ to higher values of Ro^{-1} as Ri^{-1} increases, and the drag may be much larger at high Ro^{-1} than predicted by a theory without shear.

4. DISCUSSION

The foregoing results are undoubtedly relevant as an extension of the work of Teixeira et al. (2004); however, some cautions are necessary. Both conceptually and practically, the assumptions adopted for the background wind are valid only in limited conditions. Hence, for example, it can be shown that the background wind can only be exactly stationary in the presence of shear, as assumed, if the wind is unidirectional. However, even in other situations, it is approximately stationary, since the time scale for its variation is considerably larger than the timescales typical of mountain waves (see Broad, 1995). On the other hand, the static stability of the atmosphere may only be independent of horizontal coordinates, as assumed, if the wind has a linear type of variation (see Inverarity and Shutts, 2000). Any curvature of the wind profile will imply that the static stability may not be a function of z only. Unfortunately, this assumption is not even approximately valid, as stationarity is, in a more general situation. It only holds if the product $Ri^{-1}Ro^{-1}$ is sufficiently small, which renders the present model less useful. These limitations, which are dictated by the assumption of geostrophic balance of the background wind, lead to difficulties in setting up numerical simulations of the cases treated here. For this reason, these simulations are only now being developed, and so were not included in this paper.

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