The specific LAM coupling problem seen as a filter

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Contents

Spurious reflections

1. Lateral boundary conditions in NWP are *ill-posed*
2. Numerical integration schemes ignore that, but generate *spurious* reflections at the boundaries of your domain.

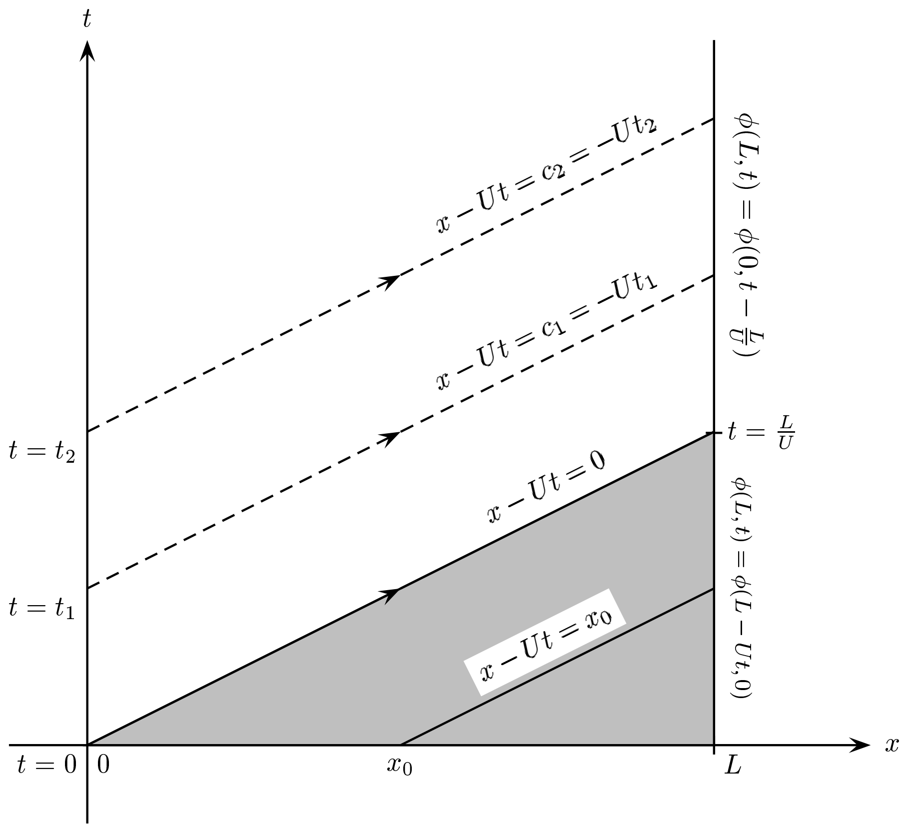
We will explain this by means of the *1D advection equation*:

- = part of the 3D model
- nice tool to illustrate
  1. simple
  2. the problem of well-posedness can be solved, so we can understand the relation between the solution of the purely mathematical problem and the numerical approach.

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} = 0$$

with $U = constant$ and on a domain $0 \leq x \leq L$.
The solution is given by

$$\phi(x,t) \equiv \phi(x - Ut)$$



Ill-pose boundary conditions

If you put LBCs at L, the right-hand-side of the boundary, then the problem is overspecified

- the continuous equation DOES NOT HAVE A SOLUTION
- the numerical scheme will not protest BUT will give *unreal* or *spurious* results

Experiment

Illustration with a leapfrog scheme:

$$\frac{\phi_i^{n+1} - \phi_i^{n-1}}{2\Delta t} = -U \, \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x}$$

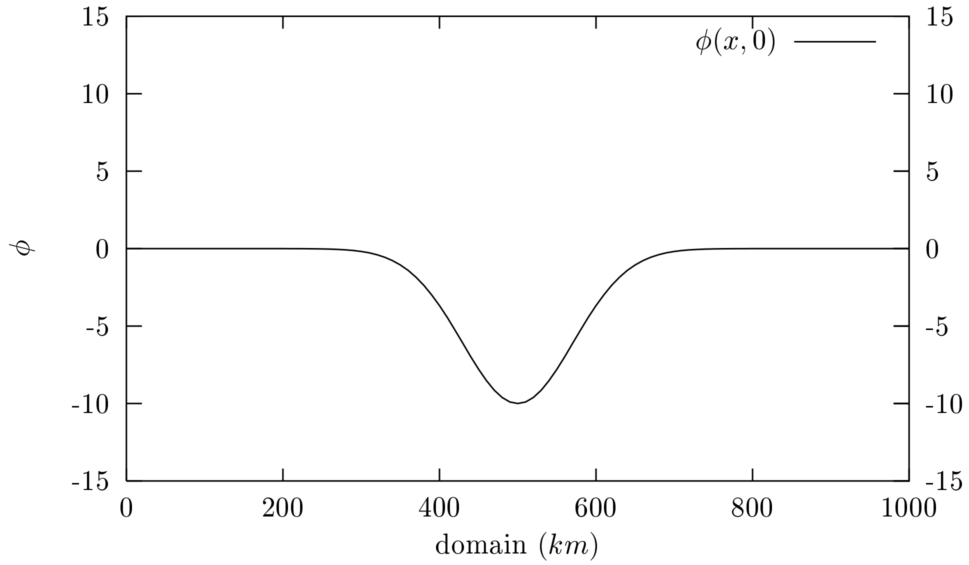
where $\phi_i^n = \phi(i\Delta x, n\Delta t)$, $U = 20m/s$

$L = 1000km$, $\Delta x = 10km, \Delta t = 50s$

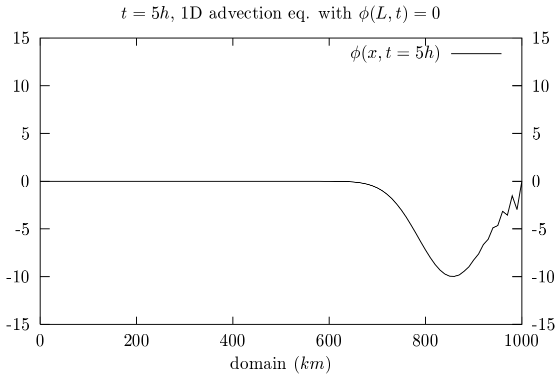
Initial condition: $\phi(x,0) = A\,\Phi(x)$

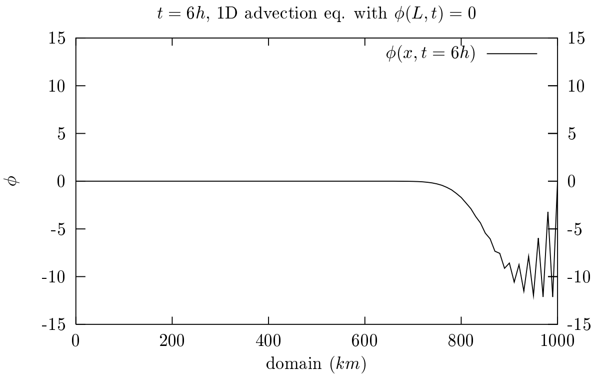
$$A = -10\,, \quad \Phi(x) = e^{-100\times\left(\frac{x-\frac{L}{2}}{L}\right)^2}$$

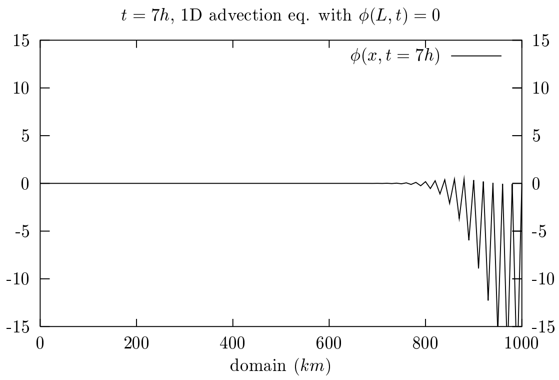
bell-shaped initial condition for the 1D advection eq.

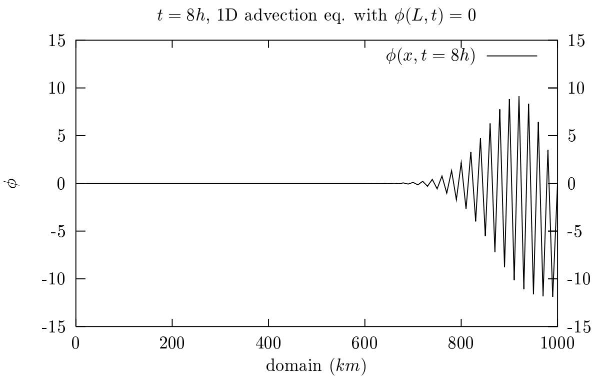


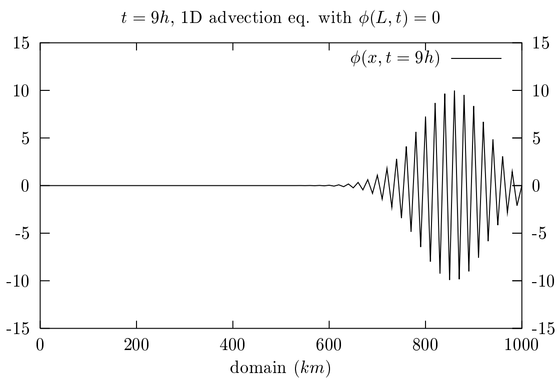
LBC: $\phi(L, t) = 0$

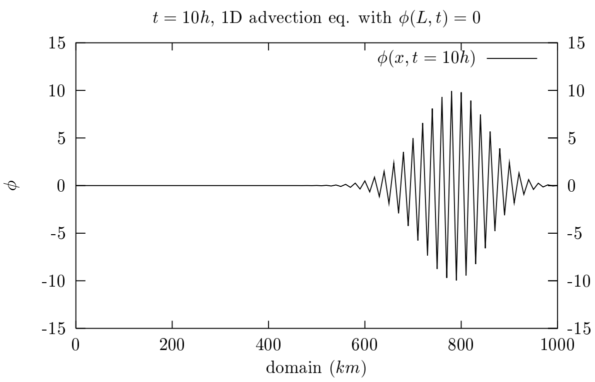

$t = 5h$, 1D advection eq. with $\phi(L, t) = 0$
$\phi(x, t = 5h)$
domain (km)


$t = 6h$, 1D advection eq. with $\phi(L, t) = 0$
$\phi(x, t = 6h)$
ϕ
domain (km)


$t = 7h$, 1D advection eq. with $\phi(L, t) = 0$
$\phi(x, t = 7h)$
domain (km)


$t = 8h$, 1D advection eq. with $\phi(L, t) = 0$
$\phi(x, t = 8h)$
ϕ
domain (km)


$t = 9h$, 1D advection eq. with $\phi(L, t) = 0$
$\phi(x, t = 9h)$
domain (km)


$t = 10h$, 1D advection eq. with $\phi(L, t) = 0$
$\phi(x, t = 10h)$
ϕ
domain (km)

Ill-posed LBCs

- We see some saw-toothed artifact being reflected

Let us try some more realistic LBCs:

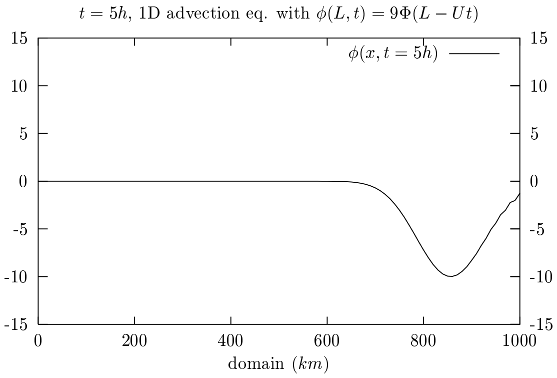
$$\phi_{LBC}(t) = B\,\Phi(L - Ut)$$

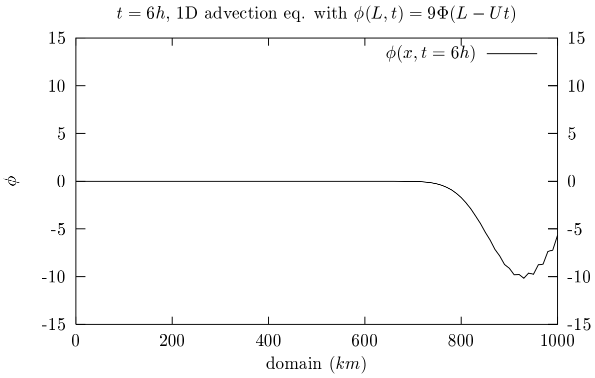
where now

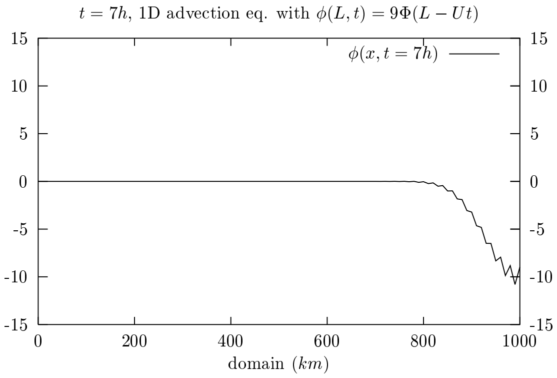
$$B = 9.$$

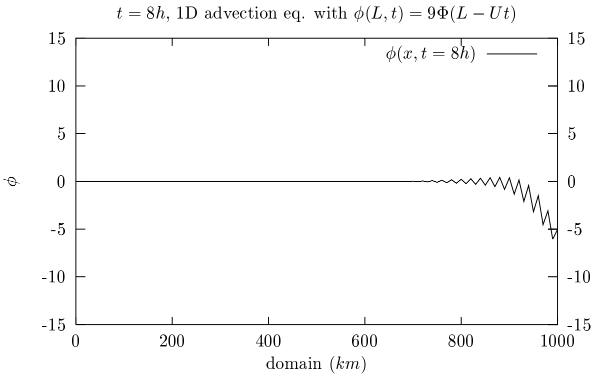
This may be representative for the way it happens in NWP. The LBCs may be given from a forecast of e.g. a global model, that underestimates the feature, due to its lower resolution.

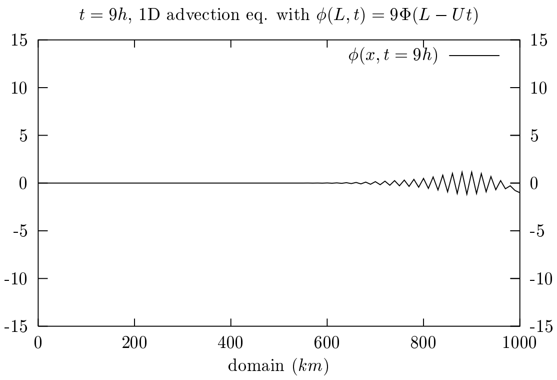
LBCs: $\phi(L,t) = 0.9\,\Phi(L - Ut)$

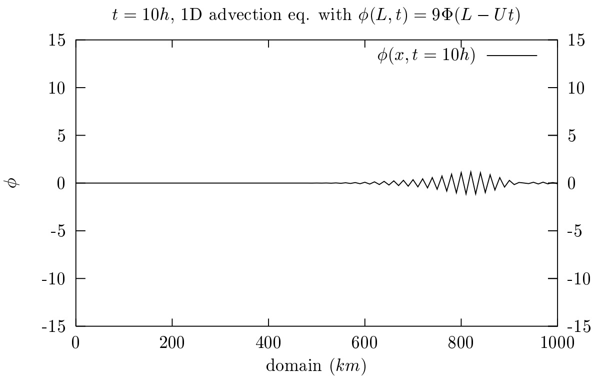












Try to put well-posed conditions

Let us try some more *realistic* LBCs:

$$\phi_{LBC}(t) = A\,\Phi(L - Ut)$$

where now

$$A = 10.$$

Even then we see some *minor* reflections:

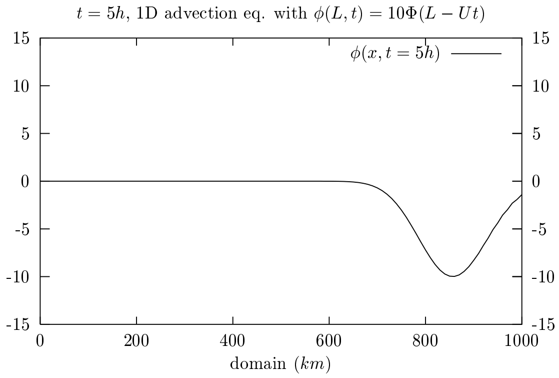
- the solution of the numerical scheme $\neq$ the solution of the continuous advection eq.
- this *small discrepancy* causes small reflections

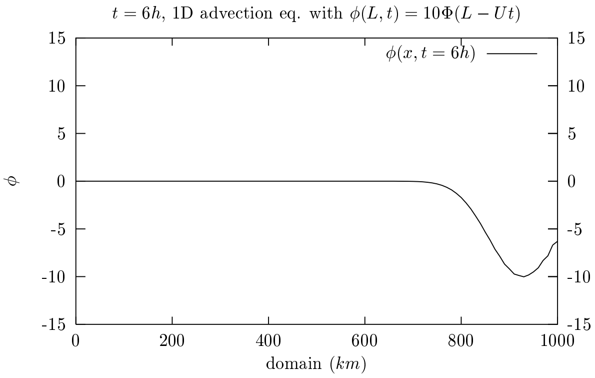
Conclusion:

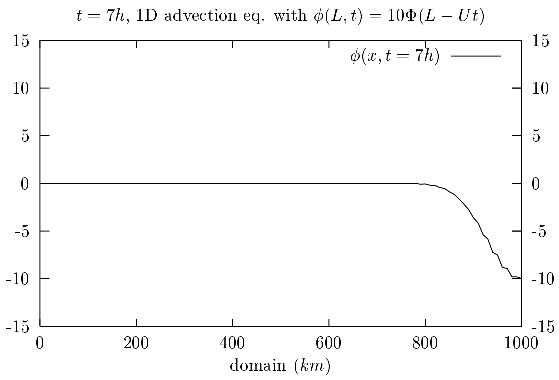
It is IMPOSSIBLE *to impose strictly well-posed lateral boundary conditions for a numerical integration scheme*

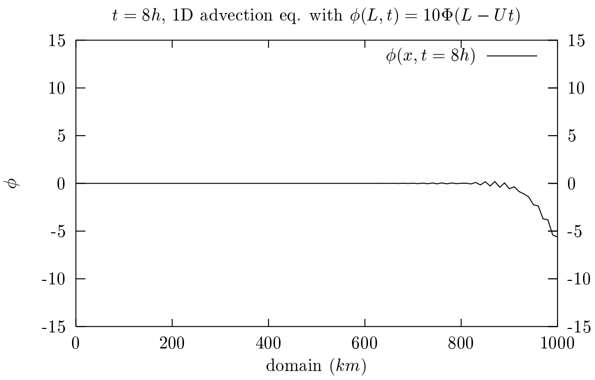
STRATEGY: filter the spurious reflections

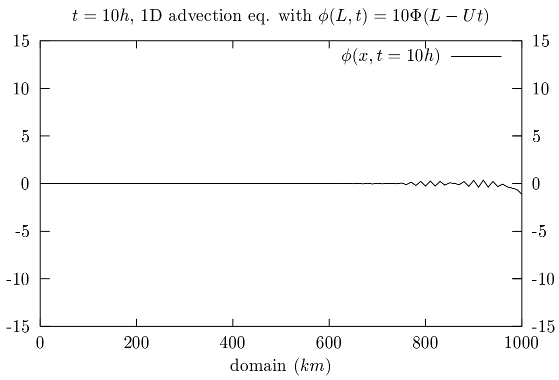
LBCs: $\phi(L,t) = 10.\,\Phi(L - Ut)$

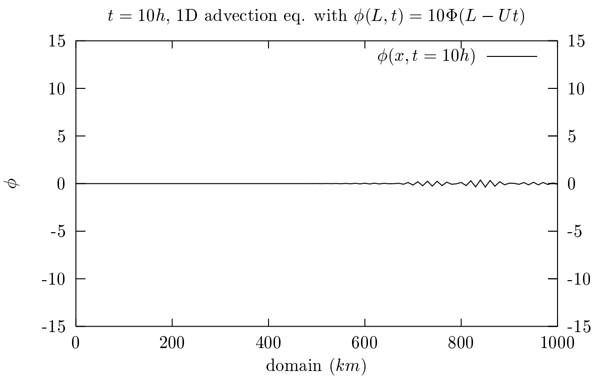












Flow relaxation

MODIFY the equations

$$\frac{\partial \phi}{\partial t}(x,t) + U\frac{\partial \phi}{\partial x}(x,t) = -K(x)\left[\phi(x,t) - \tilde{\phi}(x,t)\right]$$

$\tilde{\phi}$: is an *estimate* of the solution

$K(x)$: *non zero and positive* in a boundary zone

Take $\tilde{\phi}$ that is already (close to) a solution of the advection equation.

$$\phi' \equiv \phi - \tilde{\phi}$$

$$\frac{\partial \phi'}{\partial t}(x,t) + U\frac{\partial \phi'}{\partial x}(x,t) = K(x)\,\phi'(x,t)$$

solution:

$$\phi'(x,t) = \Phi(x - Ut)\,e^{-\frac{1}{U}\int_{x_0}^{x} K(y)\,dy}$$

$\rightarrow$ EXPONENTIAL RELAXATION in the boundary zone

relaxation DISTANCE x_* at e^{-1}: $\int_x^{x_*} K(y)\,dy = U$.

Numerical scheme

$$\frac{\phi_i^{n+1} - \phi_i^{n-1}}{2\Delta t} = -U \, \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} - K_i \, F_i^{(n)}$$

Stable solution if implicit treatment:

$$F_i^{(n)} = \phi_i^{n+1} - \tilde{\phi}_i^{n+1}$$

then

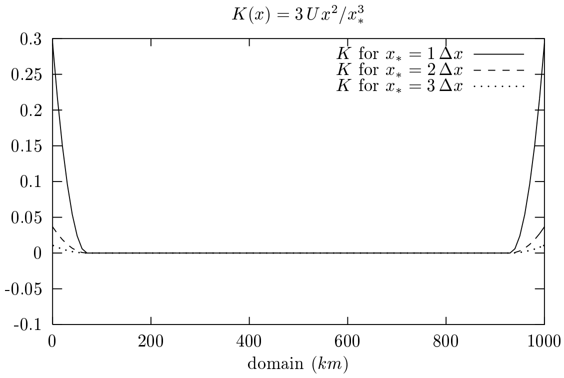
$$\begin{aligned} \phi_i^{n+1} &= (1 - \alpha_i) \left[\phi_i^{n-1} - \frac{U \Delta t}{\Delta x} \left(\phi_{i+1}^n - \phi_{i-1}^n \right) \right] \\ &+ \alpha_i \, \tilde{\phi}_i^{n+1} \end{aligned}$$

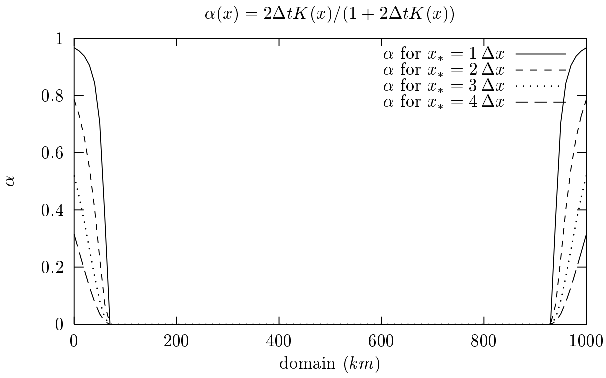
with

$$\alpha_i = \frac{2\Delta t K_i}{1 + 2\Delta t K_i}$$

Experiment

$$K(x) = \frac{3U}{{x_*}^3} x^2 \,, \qquad \frac{1}{U} \int_{x_0}^{x} K(x) = \frac{x^3}{{x_*}^3}$$

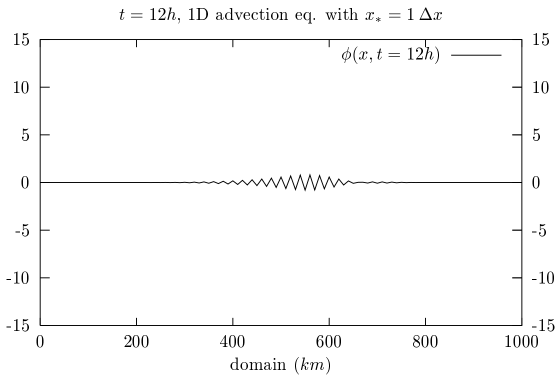


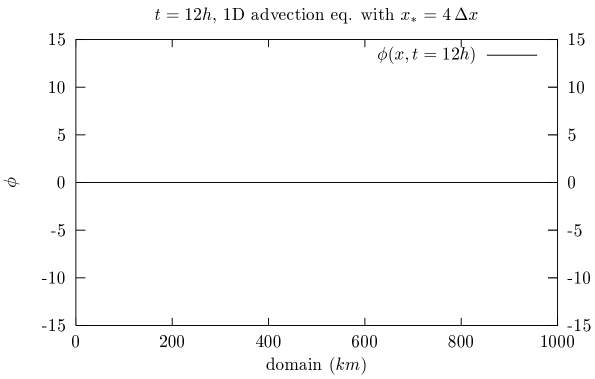


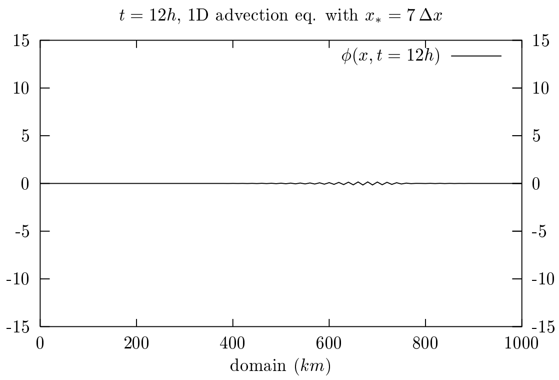
We define some sort of reflection coefficient

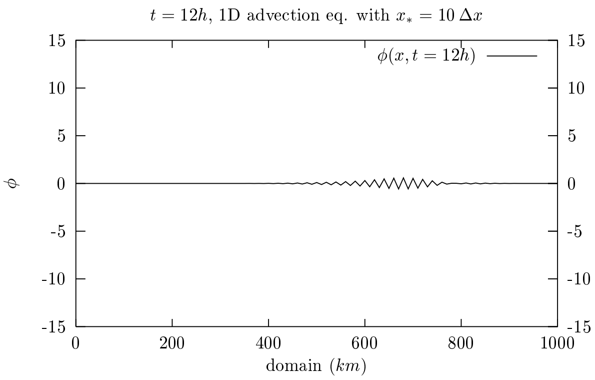
$$r = R/A \,, \qquad R = \max_{0 \le x \le L} |\phi(x, 12h)|$$

Snap shots


$t = 12h$, 1D advection eq. with $x_* = 1\,\Delta x$
$\phi(x, t = 12h)$
domain (km)

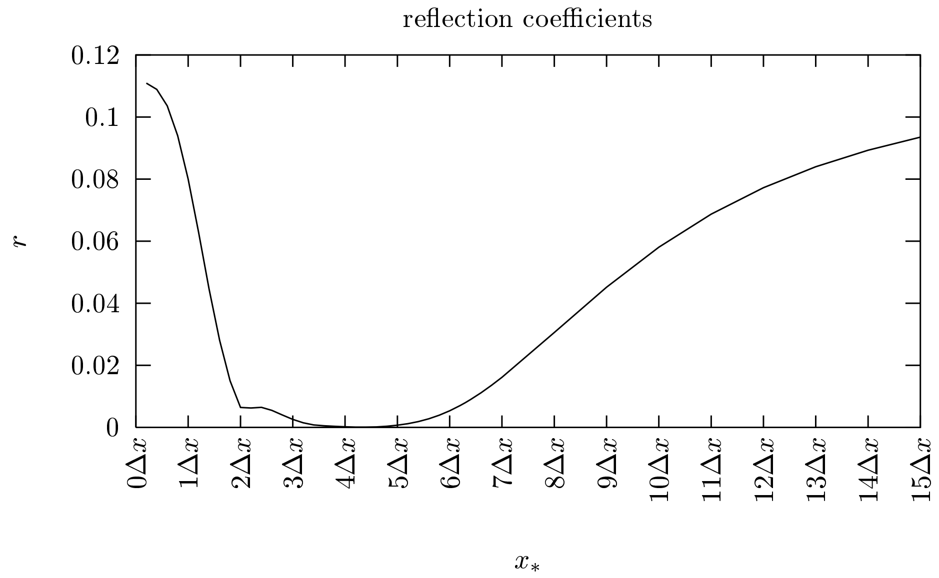

$t = 12h$, 1D advection eq. with $x_* = 4\,\Delta x$
$\phi(x, t = 12h)$
ϕ
domain (km)


$t = 12h$, 1D advection eq. with $x_* = 7\,\Delta x$
$\phi(x, t = 12h)$
domain (km)


$t = 12h$, 1D advection eq. with $x_* = 10\,\Delta x$
$\phi(x, t = 12h)$
ϕ
domain (km)

reflection coefficient as a function of x_*

Do the experiment for different choices of x_*

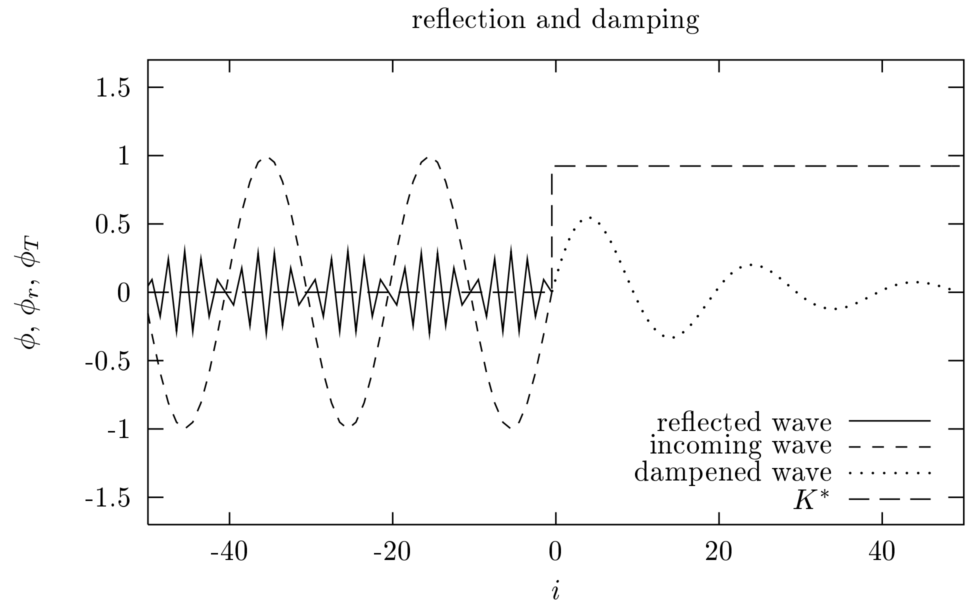


conclusion: The relaxation distance should be not too long, obviously, but not to short either!

Let us examine this more systematically (Davies 1983).

solution with reflections

summarize the situation:



change in K between $x = 0\,\Delta x$ and $x = 1\,\Delta x$

solution to the leapfrog eq.:

$$\phi(x = j\Delta x) \quad = \quad 1e^{-i(k_1 x - \omega t)} + r(-1)^j e^{i(k_1 x + \omega t)}$$

$$\phi(x = j\Delta x) \quad = \quad Te^{-\mu x} e^{-i(k_2 x - \omega t)}$$

Boundary zone of s grid points, K constant

Take a relaxation zone of s grid points where $K(j) = 0$ for $j \leq 0$ and K = *constant* in the relaxation zone where $j = 1, \ldots, s-1$.
For $j = s$ we impose $\phi(s) = \tilde{\phi}(s)$.

Call

$$K^* = \frac{K \Delta x}{U}$$

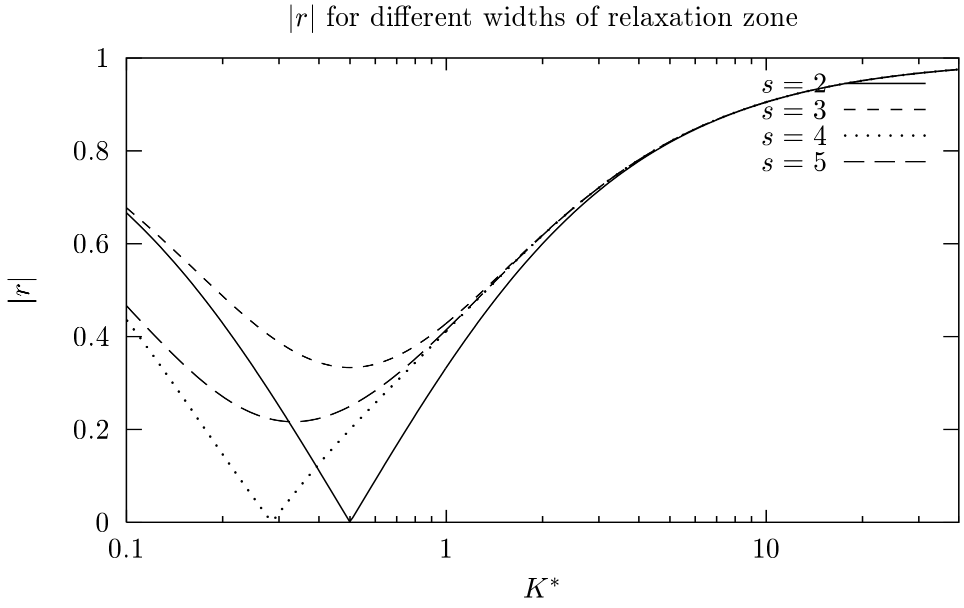
Consider the relaxation coefficient

$$r = \frac{1-\mu}{1+\mu}$$

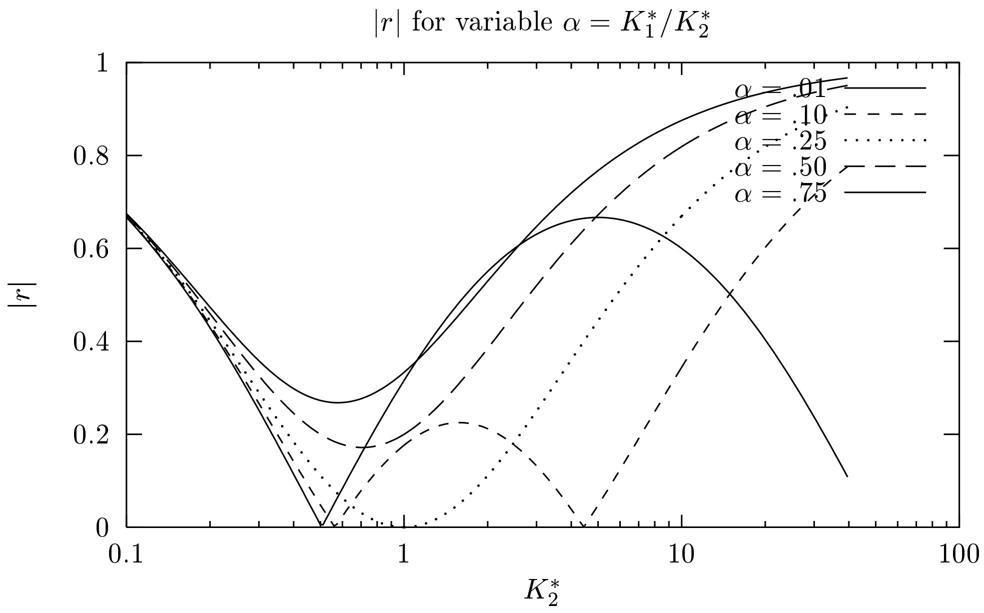
with

$$\mu = 2K^*(1) + \cfrac{1}{2K^*(2) + \cfrac{1}{2K^*(3) + \ddots \cfrac{1}{2K^*(s-1)}}}$$

Boundary zone of s grid points, K constant



Boundary zone of $s = 3$ grid points



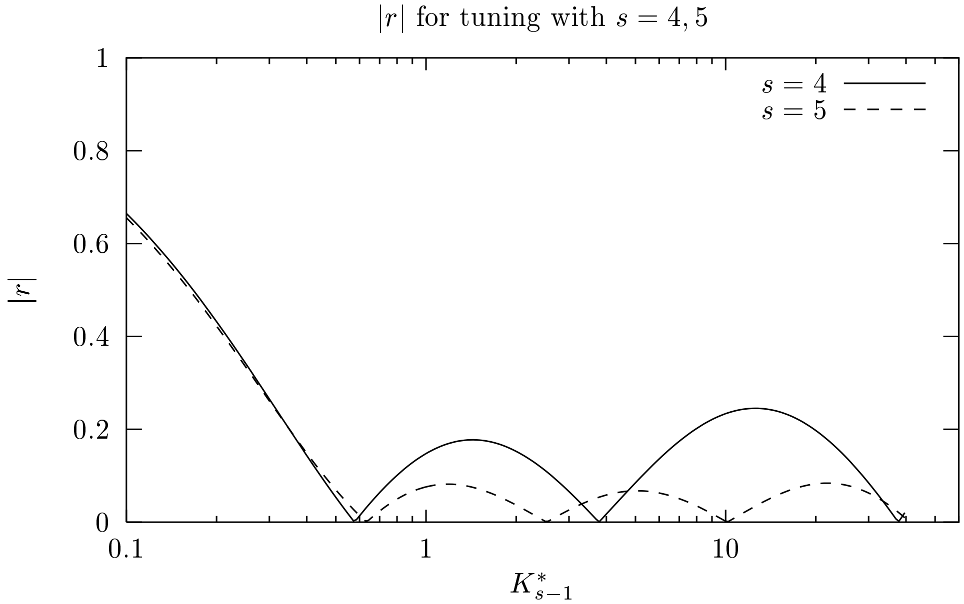
Boundary zone of $s = 4, 5$ grid points

Davies 1983:

$$s = 4: \qquad K_1^* = 0.0119\, K_3^*,\, K_2^* = 0.128\, K_3^*$$

$$s = 5: \qquad K_1^* = 0.00919\, K_4^*,\, K_2^* = 0.0491\, K_4^*,$$

$$K_3^* = 0.208\, K_4^*$$



Lehmann 1993: treat as an *optimization problem* to find optimal choice for the relaxation coefficients

Kallberg 1977: *tuning*

Particularity spectral models: Radnóti 1995

The Davies relaxation is implemented in two ways in ALADIN

- coupling done at the beginning of the time step (ELSCO):

$$\Psi = (1-\alpha)\Psi^{C} + \alpha\Psi^{LS}$$

- coupling done after grid point calculation (ELSCOT1)

$$(1-\Delta t\mathcal{L})\Psi_{t+\Delta t} = \underbrace{\Psi^{exp}_{t+\Delta t} + \Delta t\mathcal{L}(\Psi_{t-\Delta t} - 2\Psi_t)}$$

$$(1-\Delta t\mathcal{L})\,\Psi_{t+\Delta t} =$$

$$(1-\alpha)\overbrace{\left[\Psi^{exp}_{t+\Delta t} + \Delta t\mathcal{L}(\Psi_{t-\Delta t} - 2\Psi_t)\right]}$$

$$+\alpha\,(1-\Delta t\mathcal{L})\,\Psi^{LS}_{t+\Delta t}$$

Advantage: don't have to do grid point calculations in the Extension zone.

Janiskova 1994: tuning

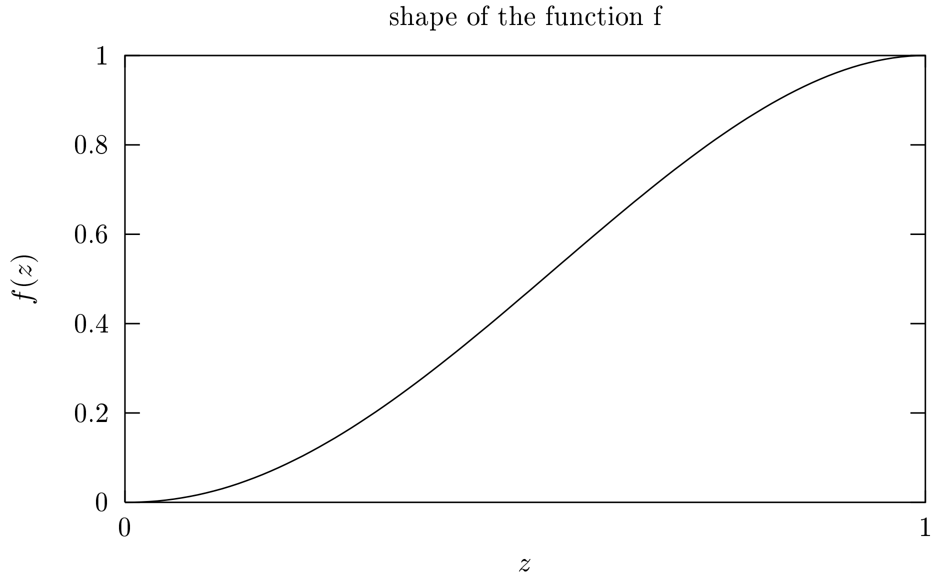
- Tuning with 1D shallow water model
- initialization with:

$$\phi(x) = \bar{\phi} + \phi_A \cos \frac{2\pi x}{L}$$

- α constructed by the funtions defined on $0 \leq z \leq 1$:

$$\begin{aligned} p \geq 2 \quad &: \quad f(z) = (p+1)z^p - pz^{p+1} \\ p \leq -2 \quad &: \quad f(z) = 1 - (p+1)(1-z)^p + p(1-z)^{p+1} \end{aligned}$$

- *tune* p to get a minimal reflection
- $\Rightarrow$ 8 gridpoints
  and $p = 2.16$ and $p = 5.52$ for humidity.


shape of the function f
1
0.8
0.6
0.4
0.2
0
$f(z)$
0
1
z

Discussion

- LBCs are always ILL-POSED in NWP
- this causes SPURIOUS REFLECTIONS
- we explained how they are dampened by the Davies scheme
- we gave some details about ALADIN