Modification of ARPEGE necessary to include the EGA-SA, "case (ii)"

Pierre Bénard
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file : EGA_Case2.tex

1 Introduction

In another memorandum (file: Continuity_EGA.tex), the form of the continuity equation in mass-based coordinates for ellipsoidal geometries has been examined. Compared to the currently used spherical geopotential approximation (SGA), the ellipsoidal geopotential approximation (EGA) would allow a reduction of geopotential errors by two orders of magnitude and this reduction could improve long-range predictions. The errors in SGA mainly consist in geometric errors and gravity errors. In Bénard, QJRMS, 2014b, there was an indication that gravity errors of SGA represents the main part of the total error of SGA and that errors were not compensative. In Bénard, QJRMS, 2014a, 2014b, it was argued that gravity errors of the spherical geopotential approximation (SGA) could be assessed in a rather simple framework, requiring minimal modifications of the current ARPEGE model, and termed “case (ii)”. This framework consists in combining the current shallow-atmosphere hydrostatic option with an EGA context but still with a spherical planet. That is, the planet is spherical, only its outer geopotentials are ellipsoidal. This therefore allows a correct representation of the gravity, and since the horizontal metrics is still the spherical one, most of the model remains valid without change, and namely, the spectral representation with *spherical* harmonic functions still may be used (if the planet was ellipsoidal, then *spheroidal* harmonic functions would have to be used, but transform algorithm to or from these functions are not known).
The present memo examines in detail what has to be changed in ARPEGE, in order to make possible simulations in this “case (ii)” framework.

2 Continuity equation in “case (ii)" of B14a

The case (ii) is a special case of non-spherical geometry which consists in a spheroidal geometry (geopotential surfaces are assumed to be spheroids) but with a spherical planet (the gravity then still may have a varying meridional profile). When this case is combined with the shallow-atmosphere approximation and the reference level for the metric is the planet’s surface one, the horizontal metric is the spherical one. One may choose the longitude λ, latitude φ and geopotential as the (ξ_1, ξ_2, ξ_3) system.
N.B.: In all this paper, we adopt a "traditional meteorological" convention for the geopotential. The geopotential is assumed to *increase* with height, while in broader "astronomic" contexts, the geopotential is commonly assumed to decrease with height (in order to asymptotically reach zero at infinite). Hence we define the geopotential ϕ here as the opposite of the scalar potential field from with the vector gravity field is derived.
Due to the axial symmetry all metric factors and gravity are independent of ξ_1, and due to the shallow-atmosphere approximation, they are also independent of ξ_3:

$$\begin{aligned} h_1 &= h_1(\varphi) \\ h_2 &= h_2(\varphi) \\ h_3 &= h_3(\varphi) \\ g &= g(\varphi), \end{aligned}$$

In the general case for ξ_3, we have

$$g h_3 = \frac{\mathrm{d}\phi}{\mathrm{d}\xi_3}, \tag{1}$$

but with the additional setting $\xi_3 = \phi$ adopted here, this therefore reduces to

$$g(\varphi)\, h_3(\varphi) = 1 \tag{2}$$

Another consequence of the choice of ϕ as ξ_3 is that

$$\frac{\partial \pi}{\partial \phi} = -\rho \tag{3}$$

The assumed spherical shape of the planet combined with the shallow-atmosphere approximation, assuming that the reference level of the metric is the ground, leads to:

$$\begin{aligned} h_1 &= a \cos\varphi \\ h_2 &= a \\ h_3 &= 1/g(\varphi) \\ h &= (a^2 \cos\varphi)/g(\varphi). \end{aligned} \tag{4}$$

The physical components of the wind are given by

$$\begin{aligned} u &= (a\cos\varphi)\,\dot{\lambda}, \\ v &= a\dot{\varphi}. \end{aligned}$$

The total time-derivative of a scalar field ψ writes

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial \psi}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial \psi}{\partial \lambda} + \frac{v}{a}\frac{\partial \psi}{\partial \varphi} + \dot{\eta}\frac{\partial \psi}{\partial \eta} \tag{5}$$

All notations in this memorandum are as in the abovementioned one (file: Continuity_EGA.tex).

2.1 Hybrid hydrostatic-pressure terrain-following coordinate η

In η coordinate, the continuity equation writes

$$\frac{\partial m}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}(mu)_\eta + \frac{1}{a}\frac{\partial}{\partial \varphi}(mv)_\eta + \frac{\partial}{\partial \eta}(m\dot{\eta}) - m\frac{v}{a}\left(\tan\varphi + \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) = 0. \tag{6}$$

2.2 Consequences for ARPEGE

This equation may be rewritten in a form similar to the one used in ARPEGE:

$$\frac{\partial m}{\partial t} + \frac{1}{a\cos\varphi}\left[\left(\frac{\partial mu}{\partial \lambda}\right)_\eta + \left(\frac{\partial mv\cos\varphi}{\partial \varphi}\right)_\eta\right] + \frac{\partial}{\partial \eta}(m\dot{\eta}) - m\frac{v}{a}\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) = 0. \tag{7}$$

This equation is identical to the one of the normal (SGA-SA) ARPEGE except the last term, which is new, and must be taken into account when running the case(ii), even in shallow-atmosphere context. Denoting by a special symbol the current (SGA-SA) divergence of $m\mathbf{V}_H$, the continuity equation in case-(ii) may be rewritten.

$$\frac{\partial m}{\partial t} + \overset{\text{SGA-SA}}{\nabla}.(m\mathbf{V}_H) + \frac{\partial}{\partial \eta}(m\dot{\eta}) - m\frac{v}{a}\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) = 0. \tag{8}$$

where

$$\overset{\mathrm{SGA-SA}}{\nabla}.(m\mathbf{V}_H) = \frac{1}{a\cos\varphi}\left[\left(\frac{\partial mu}{\partial\lambda}\right)_\eta + \left(\frac{\partial mv\cos\varphi}{\partial\varphi}\right)_\eta\right] \tag{9}$$

The simplest way to modify ARPEGE code in order to allow this case, would be to change the definition of the horizontal divergence of $m\mathbf{V}_H$. The new divergence would be the true horizontal divergence in the considered case : EGA-SA-(ii):

$$\overset{\mathrm{(ii)}}{\nabla}.(m\mathbf{V}_H) = \frac{1}{a^2\cos\varphi}\left[\left(\frac{\partial mau}{\partial\lambda}\right)_\eta + \left(\frac{\partial mav\cos\varphi}{\partial\varphi}\right)_\eta\right] - m\frac{v}{a}\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) \tag{10}$$

With this new definition, the continuity equation writes

$$\frac{\partial m}{\partial t} + \overset{\mathrm{(ii)}}{\nabla}.(m\mathbf{V}_H) + \frac{\partial}{\partial\eta}(m\dot{\eta}) = 0. \tag{11}$$

which is formally identical to the current form. Once this modification is performed the continuity equation is used under various vertical integral forms, instead of this local form. In a deep-atmosphere framework, one would have to ask which kind of vertical integral is involved (inside a pseudo-conic volume with vertical lateral boundaries, or inside a volume of unit-area horizontal section?). In the shallow-atmosphere framework, this ambiguity does not exist since the area of horizontal sections of a volume with vertical lateral boundaries is constant in height (because of the invariance of the horizontal metric with height). Consequently, the following notation is not ambiguous, even for a scalar field ψ which may exhibit horizontal variations:

$$\int_{\eta_1}^{\eta_2} \psi(\lambda,\varphi,\eta)\mathrm{d}\eta \quad \rightarrow \quad \text{not ambiguous in SA} \tag{12}$$

Consequently, the algebraic and computer processes involved in various integral forms of the continuity equation do not need to be modified. The only modification required is therefore in the definition of the horizontal divergence of $m\mathbf{V}_H$. We have, as usual:

$$\frac{\partial\pi_s}{\partial t} + \int_0^1 \overset{\mathrm{(ii)}}{\nabla}.(m\mathbf{V}_H)\,d\eta = 0. \tag{13}$$

$$m\dot{\eta} = B\int_0^1 \overset{\mathrm{(ii)}}{\nabla}.(m\mathbf{V}_H)\,d\eta - \int_0^\eta \overset{\mathrm{(ii)}}{\nabla}.(m\mathbf{V}_H)\,d\eta' \tag{14}$$

$$\dot{\pi} = \frac{u}{a\cos\varphi}\frac{\partial\pi}{\partial\lambda} + \frac{v}{a}\frac{\partial\pi}{\partial\varphi} - \int_0^\eta \overset{\mathrm{(ii)}}{\nabla}.(m\mathbf{V}_H)\,d\eta \tag{15}$$

3 Other equations

In η coordinate, the geopotential is required as a diagnostic, and is given by

$$\phi = \phi_s + \int_\eta^1 \frac{mRT}{\pi}\mathrm{d}\eta. \tag{16}$$

The convention adopted here for the definition of ϕ is the meteorological one, where ϕ increases with height. The geopotential is therefore easy to access, since in shallow-atmosphere the denominator of the integrand is simply π, not p.

The momentum equations [WW12's (A19) and (A.20)] are, in (λ,φ,ϕ) coordinates:

$$\frac{\mathrm{d}u}{\mathrm{d}t} - uv\frac{\tan\varphi}{a} - (2\Omega\sin\varphi)v = -\frac{1}{a\cos\varphi}\frac{1}{\rho}\left(\frac{\partial\pi}{\partial\lambda}\right)_{\phi} \tag{17}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + u^2\frac{\tan\varphi}{a} + (2\Omega\sin\varphi)u = -\frac{1}{a}\frac{1}{\rho}\left(\frac{\partial\pi}{\partial\varphi}\right)_{\phi} \tag{18}$$

Besides, we have, from the transformation rules, and $(\partial\pi/\partial\phi) = -\rho$:

$$\left(\frac{\partial\pi}{\partial\lambda}\right)_{\phi} = \left(\frac{\partial\pi}{\partial\lambda}\right)_{\eta} - \left(\frac{\partial\pi}{\partial\phi}\right)\left(\frac{\partial\phi}{\partial\lambda}\right)_{\eta} = \left(\frac{\partial\pi}{\partial\lambda}\right)_{\eta} - (-\rho)\left(\frac{\partial\phi}{\partial\lambda}\right)_{\eta} \tag{19}$$

$\Rightarrow$

$$\frac{\mathrm{d}u}{\mathrm{d}t} - uv\frac{\tan\varphi}{a} - (2\Omega\sin\varphi)v = -\frac{1}{a\cos\varphi}\left[\frac{1}{\rho}\left(\frac{\partial\pi}{\partial\lambda}\right)_{\eta} + \left(\frac{\partial\phi}{\partial\lambda}\right)_{\eta}\right] \tag{20}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + u^2\frac{\tan\varphi}{a} + (2\Omega\sin\varphi)u = -\frac{1}{a}\left[\frac{1}{\rho}\left(\frac{\partial\pi}{\partial\varphi}\right)_{\eta} + \left(\frac{\partial\phi}{\partial\varphi}\right)_{\eta}\right] \tag{21}$$

Momentum equations are formally unchanged compared to their SGA counterpart.
The thermodynamic equation remains also formally unchanged

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{R}{C_p}\left(\frac{\dot{p}}{\pi}\right) = \frac{Q}{C_p} \tag{22}$$

$\Rightarrow$

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{R}{C_p}\left(\frac{\dot{\pi}}{\pi}\right) = \frac{Q}{C_p} \tag{23}$$

4 Formal analogy with shallow-water systems

Staniforth and White, QJRMS, 2014 expressed the governing equations for the shallow-water system in non-spherical geometries, in presence of orography. In our case (ii) but for shallow-water, they would write:

$$\frac{\mathrm{d}u}{\mathrm{d}t} - uv\frac{\tan\varphi}{a} - (2\Omega\sin\varphi)v = -\frac{1}{a\cos\varphi}g\left(\frac{\partial H}{\partial\lambda}\right) \tag{24}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + u^2\frac{\tan\varphi}{a} + (2\Omega\sin\varphi)u = -\frac{1}{a}\left(\frac{\partial gH}{\partial\varphi}\right) \tag{25}$$

$$\frac{\mathrm{d}(H-H_s)}{\mathrm{d}t} + (H-H_s)\nabla.\mathbf{V}_H = 0 \tag{26}$$

where H and H_s are the height of free-surface and bottom rigid surface respectively, and

$$\nabla.(\mathbf{V}_H) = \frac{1}{a\cos\varphi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\varphi}{\partial\varphi}\right) \tag{27}$$

This form of the system does not directly compare to the one of ARPEGE. Namely the only change compared to the SGA-SA system is the meridional variation of g in the momentum equation (25), not in the continuity equation. This formal difference comes from the fact that ARPEGE expresses sources of momentum in term of geopotential instead of height. Defining the shallow-water version of the geopotential as

$$\phi = gH \tag{28}$$

we have

$$\frac{\mathrm{d}(\phi-\phi_s)}{\mathrm{d}t} = g\frac{\mathrm{d}(H-H_s)}{\mathrm{d}t} + g(H-H_s)\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}t}, \tag{29}$$

and

$$\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{v}{a}\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi} \tag{30}$$

The shallow-water system may therefore be rewritten as:

$$\frac{\mathrm{d}u}{\mathrm{d}t} - uv\frac{\tan\varphi}{a} - (2\Omega\sin\varphi)v = -\frac{1}{a\cos\varphi}\left(\frac{\partial\phi}{\partial\lambda}\right) \tag{31}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + u^2\frac{\tan\varphi}{a} + (2\Omega\sin\varphi)u = -\frac{1}{a}\left(\frac{\partial\phi}{\partial\varphi}\right) \tag{32}$$

$$\frac{\mathrm{d}(\phi-\phi_s)}{\mathrm{d}t} + (\phi-\phi_s)\left(\nabla.\mathbf{V}_H - \frac{v}{a}\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) \tag{33}$$

When using these variables, the analogy with the HPE system is clearer, the momentum equations are not formally modified, and the continuity equation must be modified by adding a term which represents the meridional advection of $\ln(g)$.

5 Coding aspects

The quantity $\overset{\mathrm{SGA-SA}}{\nabla}.(m\mathbf{V}_H)$, is the basic ingredient required for all diagnostics quantities (13)–(15) which are derived from continuity equation. As explained in section 2.2, this basic ingredient has to be modified by adding the extra last RHS term in (8), in order to get (10).

This basic ingredient is the quantity computed in `gpcty.F90` under the name `ZT_DIVDP(JROF,JLEV)` in section `"2.  Sum divergence"` of the code. Therefore we must add, just after this computation loop, a new loop for the addition of the new extra term.

The computation of the new extra term requires the availability of the latitude φ, $\mu = \sin\varphi$ and a the mean Earth radius (assumed spherical). The radius a is simply available through `yomcst.F90`, but φ, $\mu = \sin\varphi$ are more conveniently obtained by passing argument from the calling subroutine `cpg_gp.F90`. Hence two new arguments `PLAT`, `PMU` have to be passed from `cpg_gp.F90` to `gpcty.F90`. Since `gpcty.F90` is also called by `cpg5_gp.F90`, `cpg_gpb_nhgeogw.F90` and `pp_obs\pos.F90`, the calling sequence `CALL GPCTY` has to be modified accordingly therein.

For the definition of $g(\varphi)$, the following simple formula is used:

$$g(\varphi) = g_E\left[1 + \left(\frac{5m}{2} - \varepsilon\right)\sin^2\varphi\right], \tag{34}$$

See e.g. Staniforth and White, QJRMS, 2014, "The shallow-water equations in non-spherical geometry", Eq. (44) and discussion of applicability to φ. The symbol g_E represents the equatorial value of g but is not used, since we have:

$$\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi} = \frac{2C\sin\varphi\cos\varphi}{1 + C\sin^2\varphi}, \tag{35}$$

where $C = (5m/2) - \varepsilon$.

It must be taken into account that a multiplicative factor $\delta\eta$ is applied the the quantity computed in `gpcty.F90`, which is in fact:

$$\texttt{ZT\_DIVDP} = \delta\pi(\nabla\mathbf{V}_H) + \delta B(\mathbf{V}_H.\nabla\pi_s) = \delta\eta\left[\nabla(m\mathbf{V}_H)\right]. \tag{36}$$

Here δX is the difference of values between two adjacent half levels, hence valid at full levels. As a consequence of this multiplicative factor, the quantity to be added in fact to ZT_DIVDP is:

$$-\delta\eta \frac{m\mathbf{V}_H}{a}\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) = -\delta\pi\frac{\mathbf{V}_H}{a}\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right). \tag{37}$$