Mountain Waves - the construction of analytical solutions

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Introduction

- Analytical solution
  - Used to test dynamical kernel accuracy and correctness
  - Can be used as a perfect initial conditions
- We have developed the tool to construct the analytical solutions
  - Fully non-linear framework
  - Linear and nonlinear regimes
  - Hydrostatic and non-hydrostatic regimes
  - 2D and 3D solutions
- Tool as based on work of
  - Long (Tellus,1953)
  - Laprise and Peltier (JAS,1989)
  - Smith (Tellus,1980)

Long's analytical model (I)

If we assume:

- 2D framework (x,z)
- The steady state
- Incompressibility
- irotational
- Non-linearity

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$

- streamfunction

$$u=-\frac{\partial \psi}{\partial z},w=\frac{\partial \psi}{\partial x}$$

In such framework density is conserved along streamlines and following quantity is conservative:

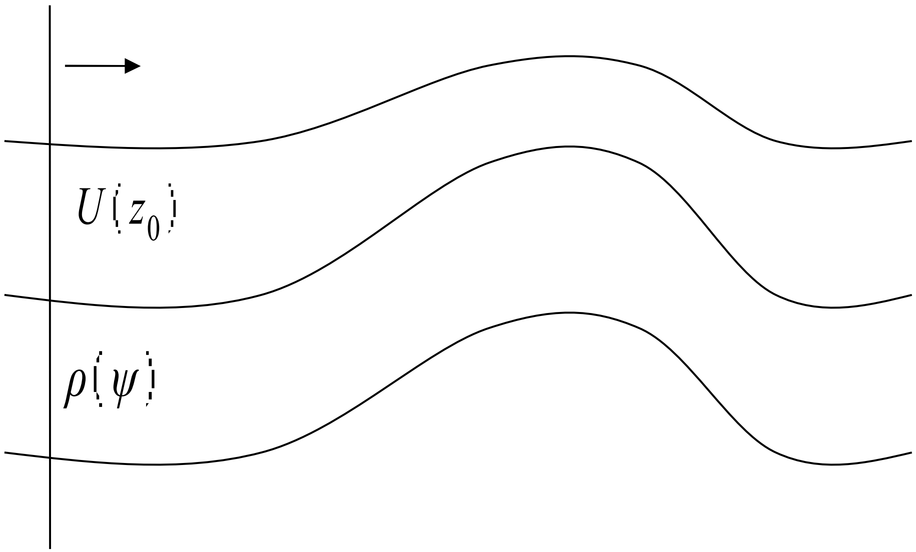
$$\nabla^2\psi+\frac{1}{\rho}\frac{\partial \rho}{\partial \psi}\left[\frac{(\nabla\psi)^2}{2}+gz\right]=const.$$

vorticity

Kinetic energy

Potential energy

Long's analytical model (II)



- The value of constant can be determined from the value of quantity far upstream
- The equation for steady-state is fully determined by the far upstream vertical distribution of wind U and density ρ(ψ)
- Static stability is conserved

$$\nabla^2\psi + \frac{1}{\rho}\frac{\partial\rho}{\partial\psi}\left[\frac{(\nabla\psi)^2}{2} + gz\right] = const. = \xi(\psi) + \frac{1}{\rho}\frac{\partial\rho}{\partial\psi}\left[\frac{U^2}{2} + gz_0\right]$$

ξ, ρ, U, z_0 are known far upstream

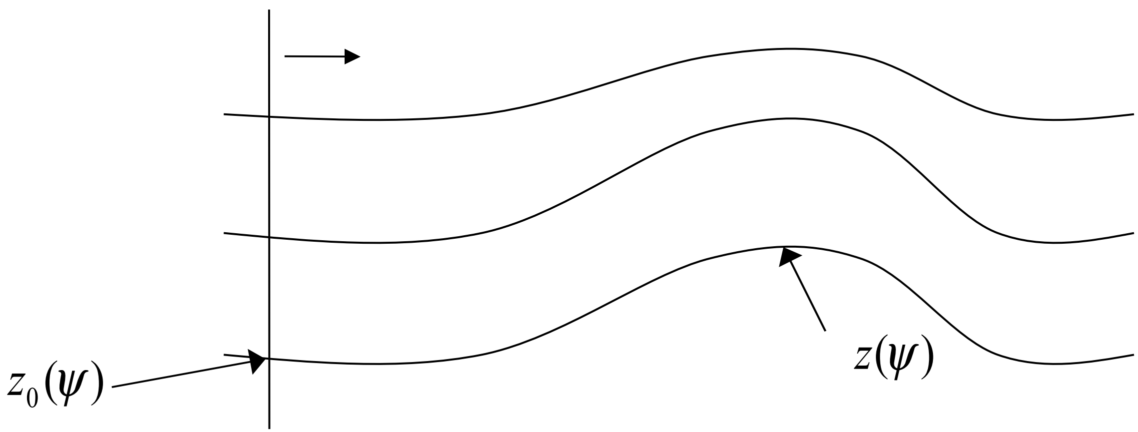
Long’s analytical model (III)

The simplifications - uniform wind and exponential decay of density.

$$U = const.$$

$$\rho = \rho_0 e^{-\beta z_0(\psi)}$$

$$\psi = U z_0$$



Introducing Perturbed stream function: $\psi' = Uz + \psi$ Equivalent to: $u = U + u'$

$$\nabla^2 \psi' + \frac{\beta}{U}\frac{\left(\nabla \psi'\right)^2}{2} - \beta \frac{\partial \psi'}{\partial z} + \frac{\beta g}{U^2}\psi' = 0$$

and:

$$\psi'(x,z) = U\left(z - z_0\right)$$

Long's analytical model (IV)

$$\nabla^2\psi' + \frac{\beta}{U}\frac{\left(\nabla\psi'\right)^2}{2} - \beta\frac{\partial\psi'}{\partial z} + \frac{\beta g}{U^2}\psi' = 0$$

For the high degree of approximation for meso-scale motions we could write:

$$\nabla^2\psi' + \frac{N^2}{U^2}\psi' = 0$$

$$N^2 = g\beta$$

BV frequency

Bottom boundary condition:

$$\psi'(x, z = h_s) = U(z - z_0) = Uh_s(x)$$

Non-linear (incompressible) steady state solution is described by the linear partial differential equation with constant coefficient.

Solution by using a FFT technique I

Technique adopted from Peltier and Laprise:

$$\psi'(x,z) = \hat{\psi}(k,z)e^{ikx}$$

Vertical structure equation:

$$\frac{\partial^2 \hat{\psi}}{\partial z^2} + (k_G - k^2)\hat{\psi} = 0$$

Hydrostatic vertical wavenumber:

$$k_G = \frac{N^2}{U^2}$$

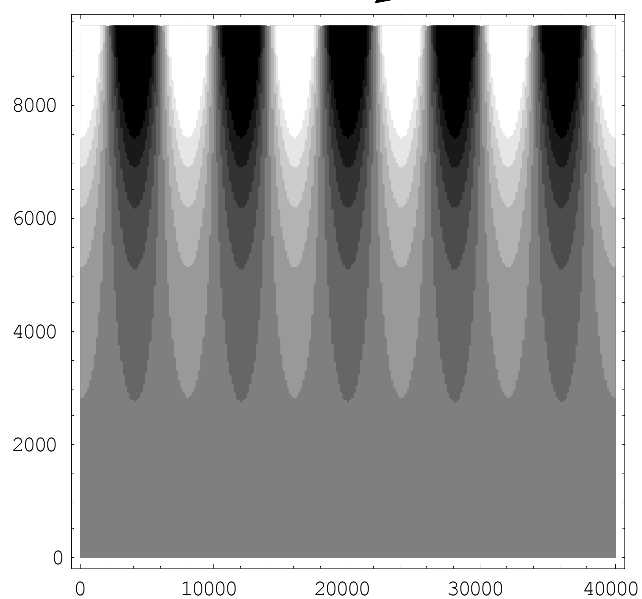
General solution in the form:

$$\psi'(x,z=0) = \sum_k \hat{\psi}(k,z=0)e^{i(kx+mz)}$$

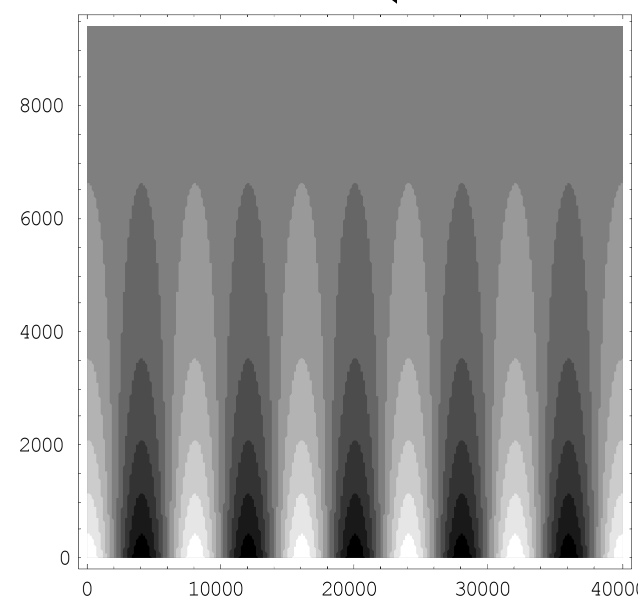
Solution by using a FFT technique II

Vertically trapped waves (horizontal short waves): $k > k_G \quad m = +- i\sqrt{(k^2 - k_G{}^2)}$

$$\psi'(x, z=0) = \sum_{k>k_G} \hat{\psi}(k, z=0) e^{ik} e^{mz} + \sum_{k>k_G} \hat{\psi}(k, z=0) e^{ik} e^{-mz}$$



Unrealistic growth with z

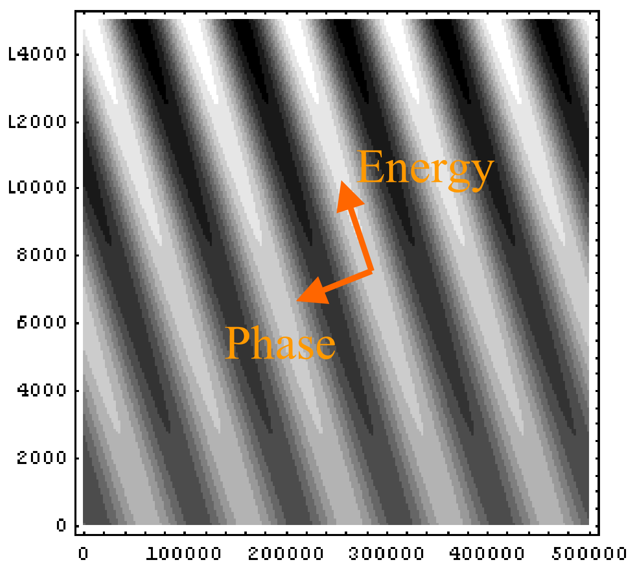


Physical mode

Solution by using a FFT technique III

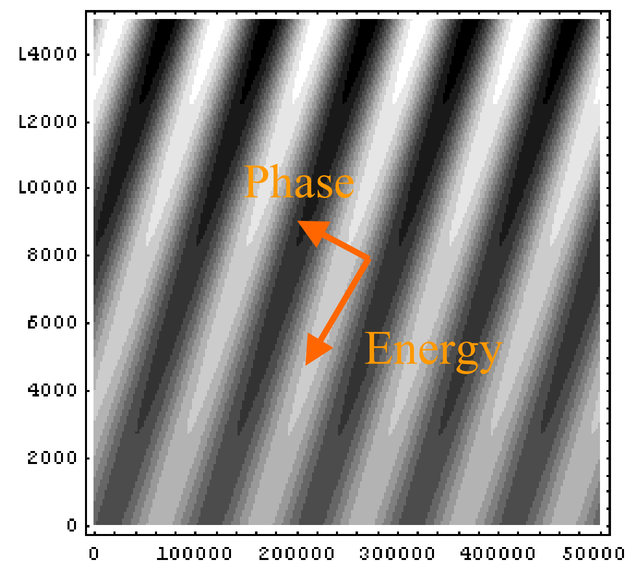
Wave modes (horizontally longer waves): $k < k_G \quad m = +-\sqrt{(k_G{}^2 - k^2)}$

$$\psi'(x,z=0) = \sum_{k>k_G} \hat{\psi}(k,z=0)e^{ik}e^{imz} + \sum_{k>k_G} \hat{\psi}(k,z=0)e^{ik}e^{-imz}$$



$$\omega^2 = \frac{N^2k^2}{k^2+m^2}$$

$$k > 0, U > 0$$

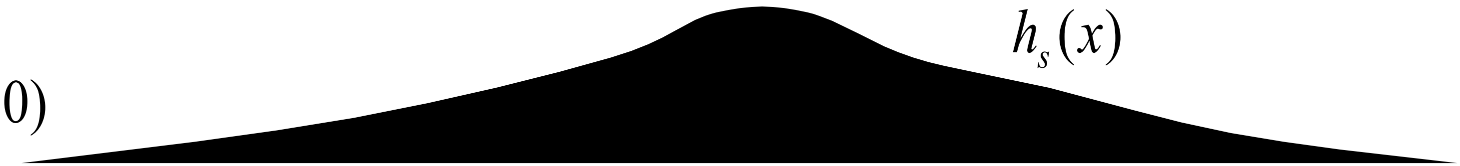


Energy is radiated in correct direction

Incorrect energy radiation

Solution – nonlinear BBC

$\psi'(x,z=0)$



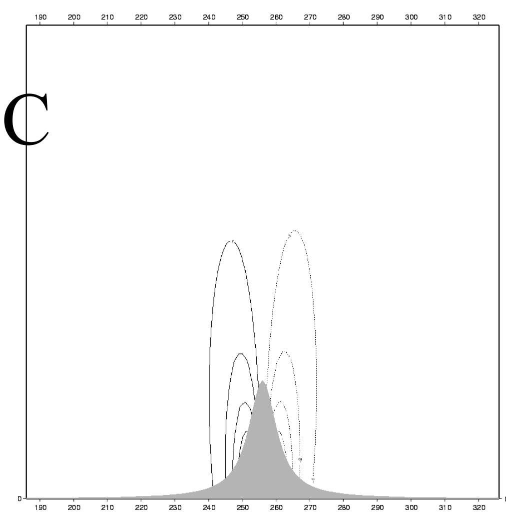
$h_s(x)$

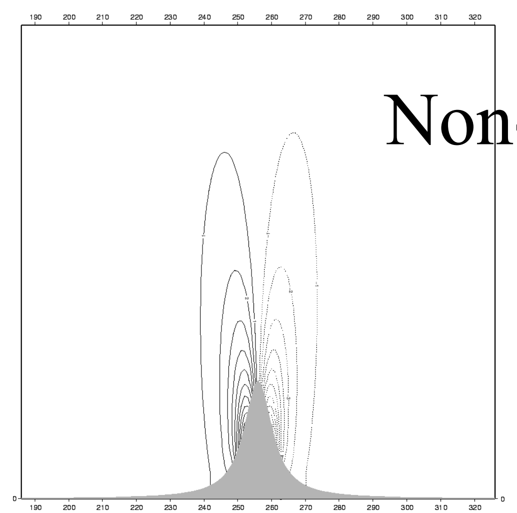
Correct BBC is implicit: $\psi'(x,z=h_s)=Uh_s(x)$

Replaced by iterative BBC:

$$\psi'(x,z=0)=\psi'(x,z=0)-\psi'(x,z=h_s)+Uh_s(x)$$

Linear BBC





Non-linear BBC

Compressibility effects

Due to compressibility the magnitude of waves increases with height. To consider this effect we assumed the isothermal atmosphere. The eigenmodes in compressible isothermal atmosphere are modulated in a following way:

$$w = w_L(z)e^{\frac{gz}{2RT_0}} \qquad u = u_L(z)e^{\frac{gz}{2RT_0}}$$

And density exponentially decreases with height:

$$\rho = \bar{\rho}e^{-\frac{gz}{RT_0}}$$

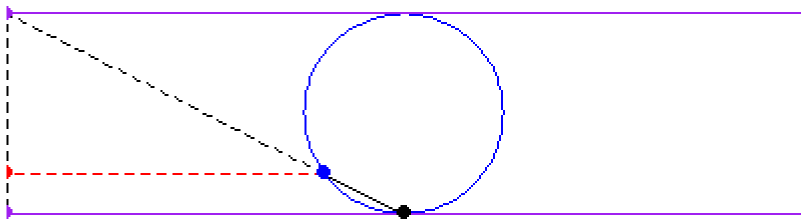
The vertical momentum flux remains unchanged comparing to pure Long's solution.

This provides reasonable results for the isothermal atmospheres. For non-isothermal atmospheres the wave patterns are correct but the increase of wave amplitudes is probably wrong (still have to be done).

Witch of Agnesi

$$h_s = \frac{H}{\left(1+\frac{x}{L}^2\right)}$$

Solutions can be constructed for any profile of surface height. But traditionally it is a so called witch of Agnesi.



For L=H

Linear analytic surf. drag:

$$drag_s = \frac{\pi}{4}\rho_0 NUH^2$$

The "witch of Agnesi" is a curve studied by Maria Agnesi in 1748 in her book *Instituzioni analitiche ad uso della gioventù italiana* (the first surviving mathematical work written by a woman). The curve is also known as cubique d'Agnesi or agnésienne, and had been studied earlier by Fermat and Guido Grandi in 1703.

The name "witch" derives from a mistranslation of the term *averisera* ("versed sine curve," from the Latin *vertere*, "to turn") in the original work as *avversiera* ("witch" or "wife of the devil") in an 1801 translation of the work by Cambridge Lucasian Professor of Mathematics John Colson (Gray).

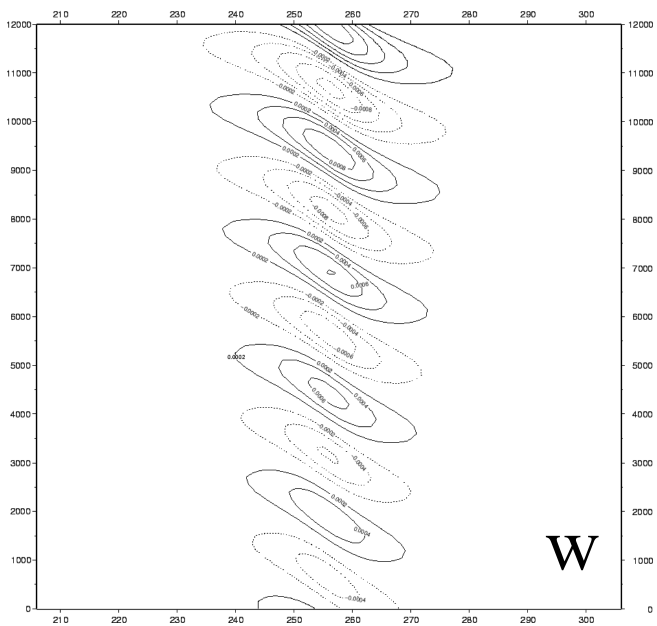
Mathworld.wolfram.com

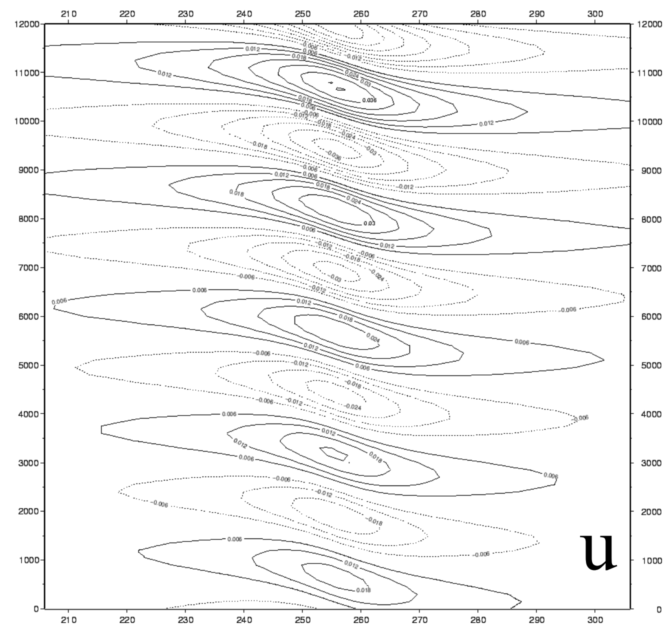
2D Linear Hydrostatic wave

Linearity scale factor: $$C_l = \frac{NH}{U} = 0.0025$$

Hydrostaticity scale factor: $$C_h = \frac{U}{NL} = 0.0025$$

Only wave modes involved, normalized vert. momentum flux:1.0



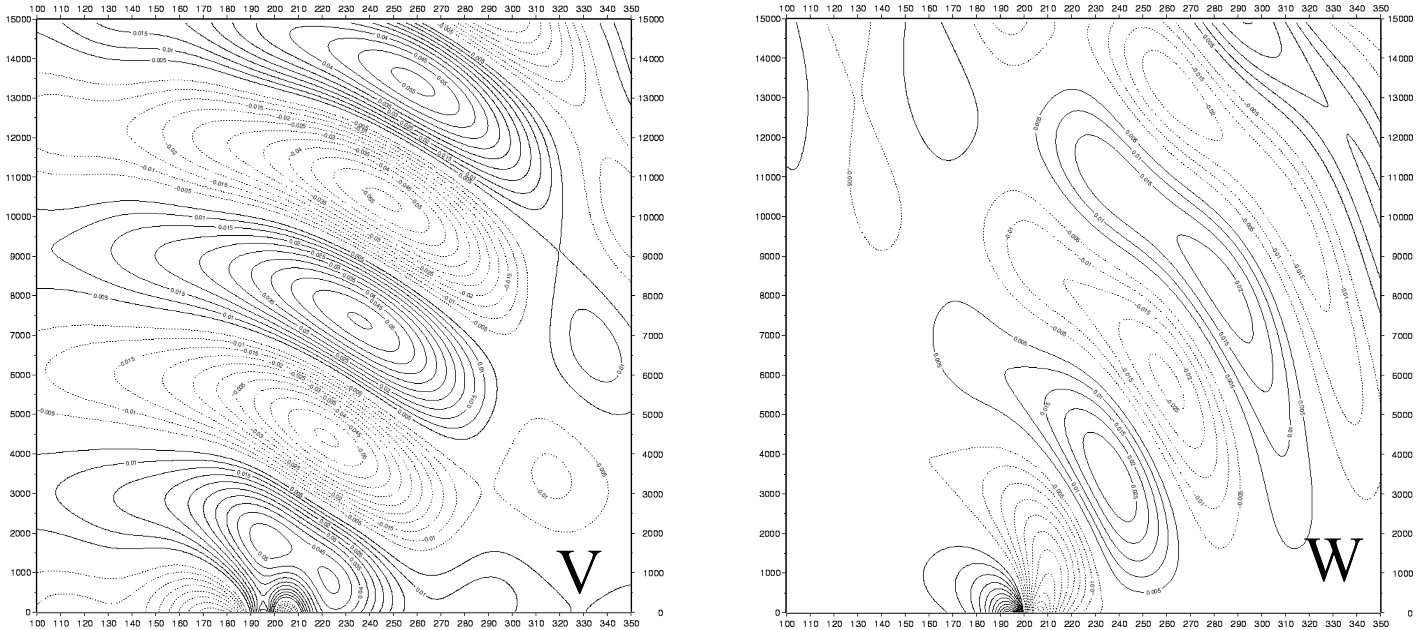


2D Linear Non-hydrostatic wave

Linearity scale factor: $$C_l = \frac{NH}{U} = 0.01$$

Hydrostaticity scale factor: $$C_h = \frac{U}{NL} = 1$$

Trapped and wave modes involved, norm. vert. momentum flux:0.45

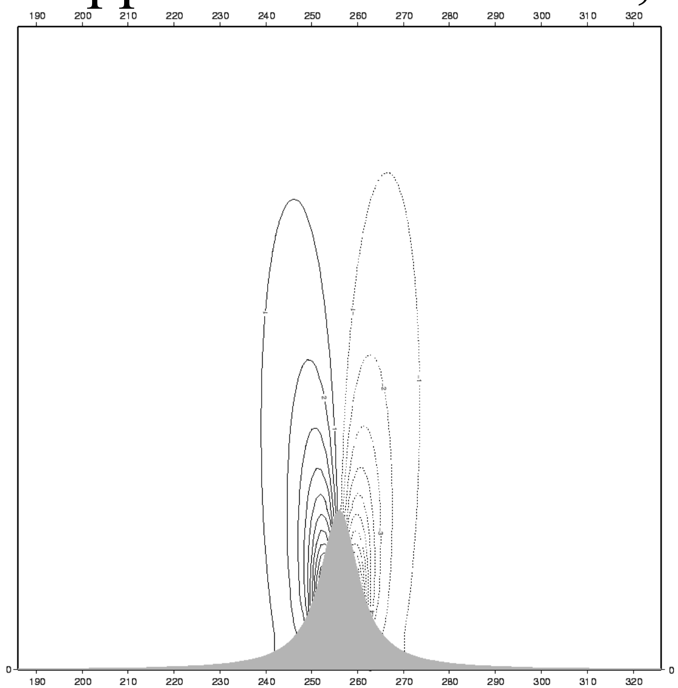


2D Potential flow

Linearity scale factor: $C_l = \frac{NH}{U} = \frac{1}{7.5}$

Hydrostaticity scale factor: $C_h = \frac{U}{NL} = 7.5$

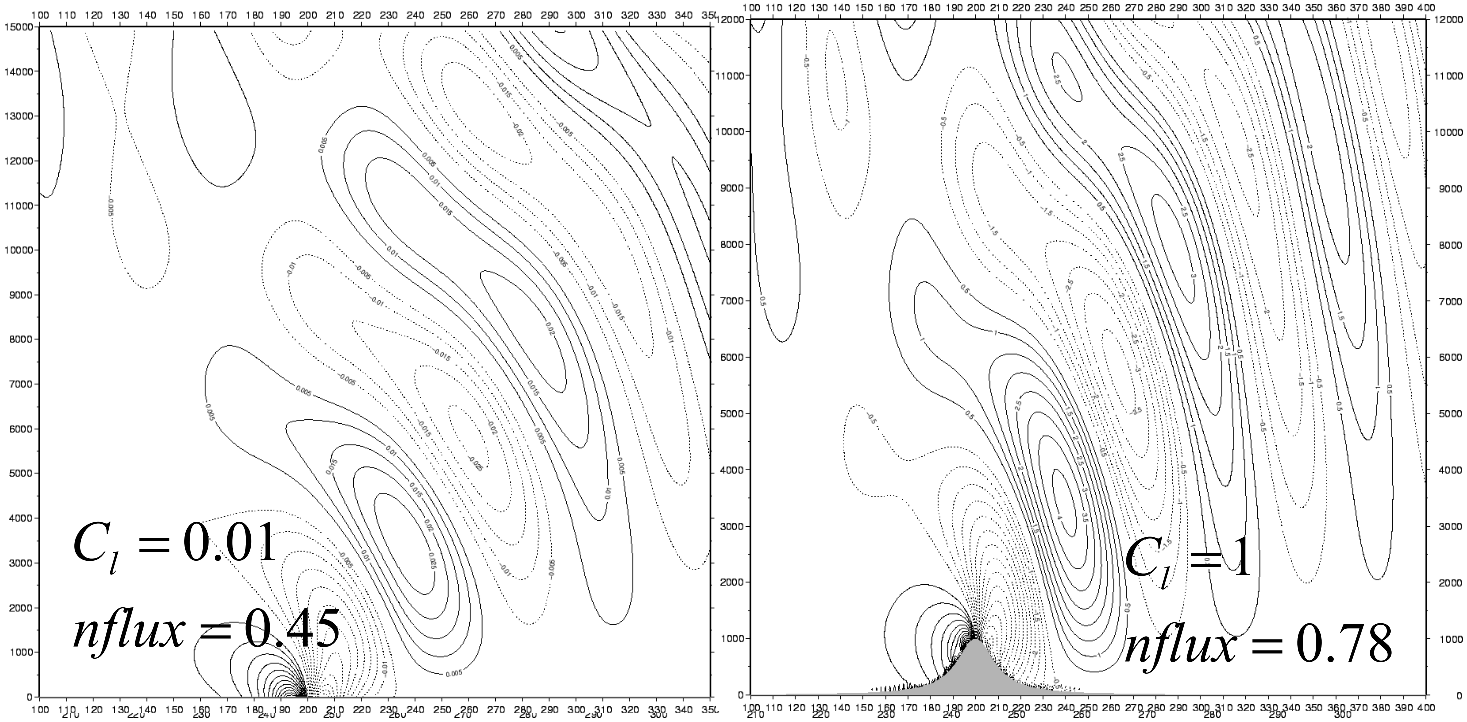
Only trapped modes involved, norm. vert. momentum flux:0.03



Very slow convergence of BBC

Non-linear flow regimes

- Same wave patterns as linear regimes
- Greater amplitude of waves
- Solution stops to be valid when u=0ms-1 occurs



Validity of Long model solution

•Valid for regimes when u=0ms-1 doesn't occur in the domain
•For regimes where u=0ms-1 occurs the gravity waves breaking associated with turbulence will occur

$$C_l = \frac{NH}{U} = 0.85$$

•Limit value for hydrostatic regimes

$$C_l = \frac{NH}{U} > 0.85$$

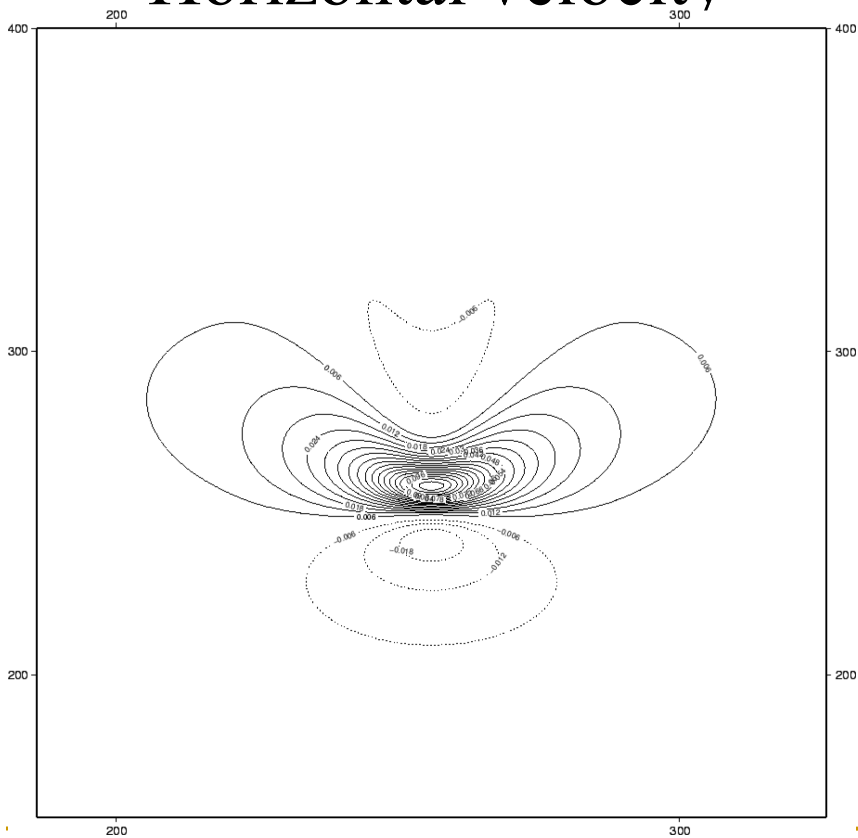
•narrower mountain can be higher before wave breaking occurs

•valid if scales of motions are less than 100km, because rotation is not taken into account

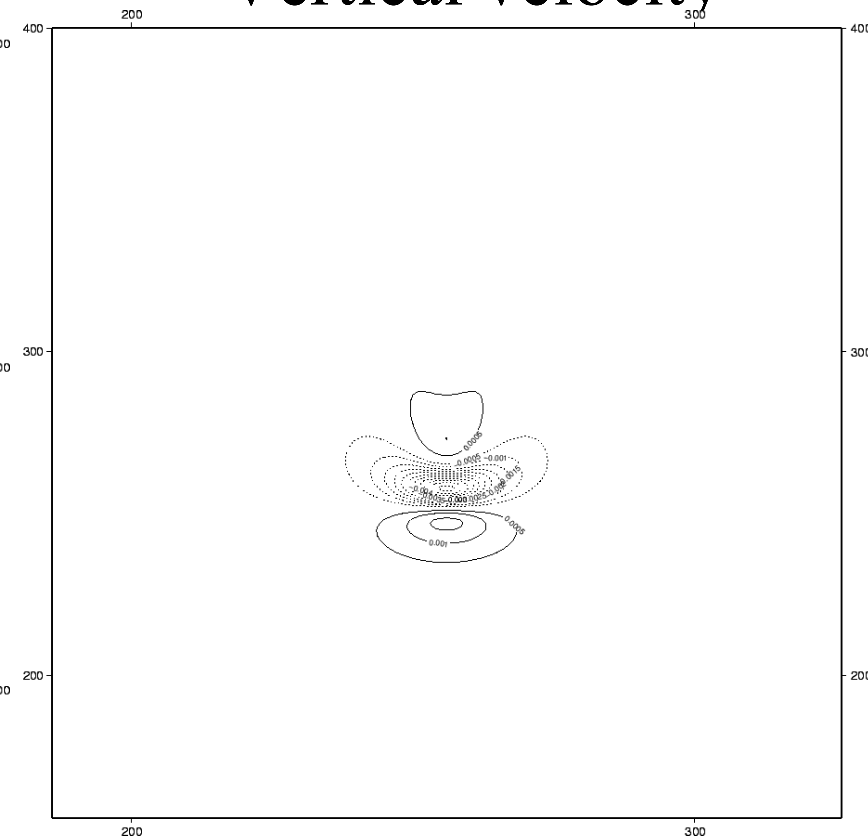
3D Hydrostatic flow

Horizontal cross-section at height $$z = \frac{1}{6}\frac{2\pi}{k_G}$$

Horizontal velocity



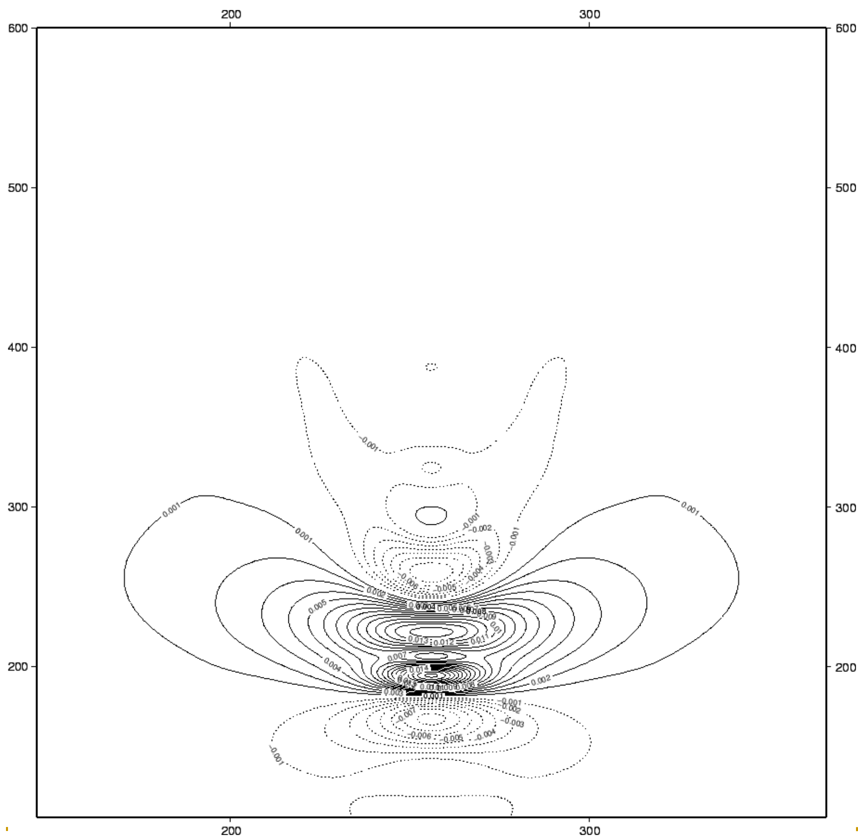
Vertical velocity



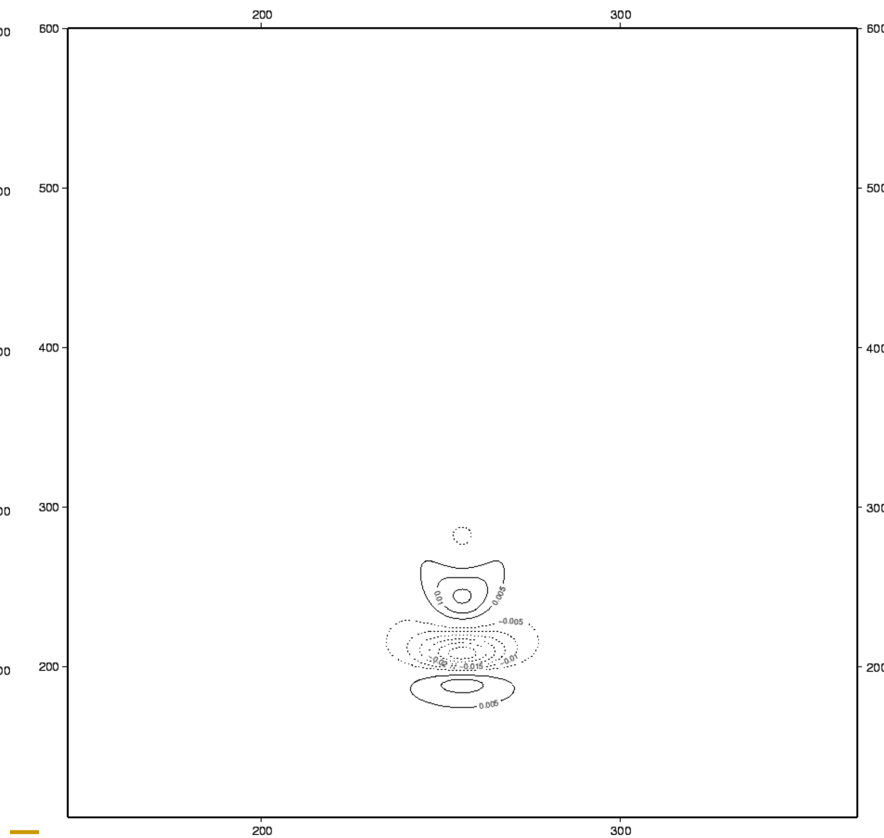
3D Non-Hydrostatic flow

Horizontal cross-section at height $z = \frac{1}{6}\frac{2\pi}{k_G}$

Horizontal velocity



Vertical velocity



At the end

You are welcome to ask for the tool on e-mail address :
Jozef.vivoda@shmu.sk
and start to validate your changes in dynamics using its results.