Improved Representation of Boundary Layer

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1 Introduction

Eddies (irregular swirls) of motion are observed at gust winds. Their size can vary from millimeters to kilometers and it is directly connected with time scale of these processes. For turbulence study and analysis, discrete forms of Navier-Stokes equations is useful. Solving of this problem is too complicate for an analytical solution. Full physics cannot be used because of computer speed limitations. The main aim is to construct the simplest possible form of equations for parameterization, but physically balanced with reality. Absence of parameterization deteriorates the numerical predictions. From experiences is apparent that the resolution of the model and atmospheric stability are factors, which gain the form of parameterizations. The goal is to find appropriate turbulence scheme which can be apply to the the problems with anisotropy as well as to the free stream air turbulence.

The energy spectrum of turbulence is not homogeneous at reality. Nevertheless, we often except, that turbulence is a random process with independence on position, what is a definition of homogeneous turbulent process. Away from boundaries and from a stable stratified atmosphere this assumptions work correctly. Resolved eddies are isotropic and small energetics eddies can cascade until viscosity dissipation. Anisotropy is assumed in other cases. Near the wall, the size of energetic eddies becomes proportional to the distance from this barrier. Increasing of the resolution in the vertical direction does not help to solve anisotropy problem, but it creates a problem with grid anisotropy, with numerical integration stability and still does not describe the situation correctly.

In this article we show possible methods, which have the base in application of the turbulent kinetic energy (TKE) as diagnostic and prognostic variable, for solving closure parameterization problems. K-theory is used for kinematic eddy fluxes, the exchange coefficient "K" has dependence on TKE. Louis integral stability profiles are modified and investigated. Relations with TKE are presented. Determination of mixing length and characteristic length are done with help of stationary (balanced) TKE theory. On the end of article, there are presented numerical integration schemes for prognostic second order differential TKE equation and obtained results are compared.

2 The parameterization scheme

The local physical tendencies $\left(\frac{\partial \mathbf{U}}{\partial t}\right)_\psi$ of vertical turbulent fluxes one can express now:

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)_\psi = -g\frac{\partial}{\partial p}J_\psi \tag{1}$$

We suppose, that J_ψ on the top of the model is zero and on the ground we use properties of the thin last model level for computation of the boundary condition. It mean that we have different formulas for fluxes computation in free atmosphere and on the surface.

In 1D situation, where the vertical exchanges are expected only, equation for TKE has form:

$$\begin{aligned}\frac{\partial \bar{e}}{\partial t} &= -\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\bar{W}e\right) + \frac{g}{\bar{\theta_v}}(\overline{w'\theta_v'}) - (\overline{u'w'})\frac{\partial \bar{U}}{\partial z} - (\overline{v'w'})\frac{\partial \bar{V}}{\partial z} - \\ &\quad -\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\overline{w'\left(p'+\bar{\rho}e\right)} + \frac{1}{\bar{\rho}}\overline{p'\frac{\partial w'}{\partial z}} - \epsilon \end{aligned} \tag{2}$$

where e is an TKE, g is a gravitational constant, θ_v is a virtual potential temperature, $\bar{U}, \bar{V}$ are components of mean horizontal wind, ρ is a density, p' is a pressure perturbation, u', v', w' are perturbation of momentums and ϵ is viscous dissipation term. It can be recognized a buoyant production or consumption term and shear term. On the next, pressure correlation, turbulent transport term (they present vertical diffusion) and dissipation term are described. Eddy-diffusivity assumption for the diffusive term is used, knowledges about Kolmogorov inertial subrange are apply for dissipation term. During unstable situation large buoyant term is associated with large value of this term and it allows turbulence. On contrary, during the stable situation this term is negative and consumes turbulence.

After application of parameterization one can write parameterized equations for TKE, vertical kinematic eddy heat and U,V momentum fluxes in 1D situation, where the vertical exchanges are expected only:

$$\begin{aligned}\frac{\partial \bar{e}}{\partial t} &= \frac{g}{\bar{\theta_v}}(\overline{w\theta_v}) - (\overline{uw})\frac{\partial \bar{U}}{\partial z} - (\overline{vw})\frac{\partial \bar{V}}{\partial z} - \\ &\quad -\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\bar{W}e\right) - \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(c_{2m}\bar{\rho}L_k e^{1/2}\frac{\partial e}{\partial z}\right) - c_\epsilon \frac{e^{3/2}}{L_\epsilon}\end{aligned} \tag{3}$$

where

$$\overline{w\theta_v} = -K_h \frac{\partial \bar{\theta_v}}{\partial z} \qquad \overline{uw} = -K_m \tfrac{\partial \bar{U}}{\partial z} \qquad \overline{vw} = -K_m \frac{\partial \bar{V}}{\partial z} \tag{4}$$

Exchange coefficients have next expressions:

$$K_h = \frac{2}{3}\frac{L_k}{c}e^{1/2}\varphi_i \quad [m^2.s^{-1}] \qquad K_m = \frac{4}{15}\frac{L_k}{c_m}e^{1/2} \quad [m^2.s^{-1}] \tag{5}$$

L_k, L_ϵ are mixing and dissipation length. Main difference between K_h and K_m is in application of function φ_i. φ_i has relation with a reciprocal turbulent Prandtl number. This function is achieved by using of balanced (stationary) approach in TKE equation. Stationary means, that in the governed equations was used a balance of shear production, buoyant and dissipation terms. One has expression for it:

$$\varphi_i = \frac{1}{\left(1 + c_3 \frac{Ri}{f(Ri)}\right)} \tag{6}$$

where function $f(Ri)$ comes from expression for stationary TKE and has form:

$$\begin{aligned} f(Ri) &= \frac{1}{2}\{1 - (c_3 + c_2)Ri \pm \\ &\quad \left.\sqrt{\left[1 - (c_3 + c_2)Ri\right]^2 + 4\left[c_3 Ri + c_3 c_2 (Ri^2)\right]}\right\}\end{aligned}$$

where Ri is Richardson bulk number. The constants $c_{2m}, c_m, c, c_\epsilon$ are a parameterized constants coming from the transport of TKE, from the pressure-correlation parameterization and from the dissipation parameterization. For additional constants one has expressions:

$$c_2 = \frac{5c_m}{2c} \qquad c_3 = \frac{5c_m c_\epsilon}{c c_0}$$

c_0 has origin in parameterization of virtual potential temperature variance perturbation.

In stationary equilibrium one can find next formulas for momentum and heat fluxes:

$$u_*^2 = \frac{e}{\alpha}\left(\frac{L_\epsilon}{L_k}f(Ri)\right)^{-1/2} \quad \text{where} \quad u_*^2 = \left(\overline{uw}^2 + \overline{vw}^2\right)^{1/2} \text{and} \quad \alpha = \left(\frac{15c_m}{4c_\epsilon}\right)^{1/2}$$

Subscript "*" marks surface fluxes.

For heat flux in alternative form towards to vertical kinematic eddy fluxes one has formulas:

$$-\overline{w\theta_v} = u_*\theta_{v*} = u_*^2 c_2 \varphi_i(Ri) Ri \frac{\theta_v}{g}\left|\frac{\partial \vec{V}}{\partial z}\right| \quad \text{or} \quad \overline{w\theta_v} = -u_*^2(1 - f(Ri))\frac{\theta_v}{g}\left|\frac{\partial \vec{V}}{\partial z}\right|$$

From knowledges and equations above one can write formulas for TKE scale (e_*), for friction velocity (u_*) and for temperature scale (θ_{v*}). In this situation, expression for parameterized stationary TKE, which contain amplitude of stress tensor, is supplied with next formula:

$$e_* = \frac{4}{15}\frac{L_\epsilon}{c_\epsilon}\frac{L_k}{c_m}\left|\frac{\partial \vec{V}}{\partial z}\right|^2 f(Ri) \tag{7}$$

In like manner we have formulas:

$$u_*^2 = \frac{4}{15}\frac{L_k}{c_m}{e_*}^{1/2}\left|\frac{\partial \vec{V}}{\partial z}\right| \qquad u_*\theta_{v*} = \frac{2}{3}\frac{L_k}{c}{e_*}^{1/2}\varphi_i\left(\frac{\partial \theta_v}{\partial z}\right) \tag{8}$$

From construction of e_* and u_* is apparent, that there is close relation between formulas (7) and (8). In simplest case, when homogeneous turbulence and neutral air stratification are assumed, we can write:

$$\widetilde{u_*}^2 = \frac{\widetilde{e_*}}{\alpha} \text{ where } \widetilde{u_*}^2, \ \widetilde{e_*} \text{ denote an neutral air stratification.} \tag{9}$$

Generally, dependence between e_* and u_* is more complicated especially in unstable case. Reason is defined TKE as $e = 0.5(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$. It is represent diagonal part of Reynolds stress tensor. Vertical fluxes handle with mixed variance and anisotropy of turbulence is usually more significant in vertical direction. Cascading of eddies to the viscous dissipation is different in horizontal than in vertical direction. For this reason we can expected different "behavior" of terms $\overline{u'^2}$ than terms $\overline{w'^2}$ or $\overline{uw}$.

First we remember the Monin Obukhov theory and Businger-Dyer or Homgstrom experimental empirical functions for wind shear and temperature gradient in the surface layer [3],[1]. For a dimensionless wind shear we use notation $\Phi_m(z/L)$ and for a dimensionless temperature profile $\Phi_h(z/L)$, where z is a height and L is the Obukhov length. Expression $\frac{K_m}{K_h}$ is the ratio of eddy diffusivityies of heat and momentum.

The ratio of heat and momentum fluxes is equals 0.74 for Businger profile and 0.95 for Homgstrom profile in neutral conditions. For next computation is convenient to introduce a surface-layer scaling parameter $\zeta = \frac{z}{L}$.

From these profiles one can construct Richardson number (Ri) dependence on surface-layer scaling parameter.

Let we introduce notation $S(Ri(\zeta))$ for next function:

$$S(\zeta) = \left(\frac{L_\epsilon(\zeta)}{L_k(\zeta)}f(Ri(\zeta))\right)^{1/2} \tag{10}$$

After substitution of function $S(\zeta)$ to the (8) one has relation for friction velocity:

$$u_* = \frac{4}{15}\frac{L_k(\zeta)}{c_m}\alpha^{1/2}S(\zeta)^{1/2}\left|\frac{\partial\vec{V}}{\partial z}\right| \quad \text{or} \quad u_* = \left(\frac{e_*}{\alpha}\right)^{1/2}S(\zeta)^{-1/2} \tag{11}$$

Analogically we obtain formula for scale temperature:

$$\theta_{v*} = \frac{2}{3}\frac{L_k(\zeta)}{c}\alpha^{1/2}S(\zeta)^{1/2}\varphi_i(Ri(\zeta))\left(\frac{\partial\theta_v}{\partial z}\right) \tag{12}$$

From these formulas we have expression for surface fluxes in system with exchange coefficients dependence by TKE.

Last requirement is to specify function $S(\zeta)$, thus we need to know an ratio $\left(\frac{L_\epsilon(\zeta)}{L_k(\zeta)}f(Ri(\zeta))\right)$. This ratio has a close connection to the ratio of vertical perturbation velocity variance and TKE.

In simplest case one has: $S(\zeta) = 1$.

For unstable case to have a correct result is necessary to introduce some stability function from observations. We need to know that measurements have limitation too. Problems are to determine the mixing length during convective situation with weak vertical gradient. From measurement one has relation [5]:

$$S(\zeta) = \begin{cases} 1 + \frac{1}{\alpha}(-\zeta)^{2/3} & for\,\frac{z}{L} < 0(unstable) \\ 1 & for\,\frac{z}{L} \geq 0(neutral\,and\,stable) \end{cases} \tag{13}$$

2.1 Eddy length scale

To close unknown one can compare described fluxes above with measurements. We can use different approaches for constructing the eddy lengths L_ϵ, L_k. We will suppose that in neutral stratify atmosphere $L_\epsilon = L_k$. In this place we try to describe the several possible methods for setting of L_ϵ, L_k. One can suppose that $L_\epsilon = L_k$ are equal not only in neutral air stratification. This is not very suitable approximation of reality, especially in LAM or GCM models.

Better assumption is to suppose that $L_\epsilon \neq L_k$. Different results in stationary TKE and eddy lengths expressions one can see at this situation. Next relations for eddy lengths can be obtained after algebraic manipulation with basic equations:

$$L_k = \alpha^{3/2}c_\epsilon L\frac{S(\zeta)^{-1/2}}{\Phi_m(\zeta)} \qquad L_\epsilon = \alpha^{3/2}c_\epsilon L\frac{S(\zeta)^{3/2}}{f(Ri(\zeta))\Phi_m(\zeta)} \tag{14}$$

where $L = \kappa(z + z_0)$ and κ is a Karman constant.

With present choice of eddy lengths setting (14) one can see, that surface momentum flux is unaltered for "ζ" dependence, for scaled virtual potential temperature one has different formula:

$$u_*^2 = \left(\frac{L}{\Phi_m(\zeta)}\right)^2\left|\frac{\partial\vec{V}}{\partial z}\right|^2 \qquad \theta_{v*} = \left(\frac{L}{R\Phi_h'(\zeta)}\right)\frac{\partial\theta_{v*}}{\partial z} \tag{15}$$

where $\frac{1}{\Phi_h'(\zeta)} = \frac{\varphi_i(\zeta)}{\Phi_m(\zeta)}$.

Now we can apply next possible setting of $S(\zeta)$ which are analyzed in this article.

1. $S(\zeta) = 1$. This case of setting is very close to the current ALADIN computation of turbulent fluxes. For comparison one can introduce function from observations instead theoretical computed function $\varphi(Ri(\zeta))$. We can use substitution:

$$\varphi(Ri(\zeta)) = \frac{\Phi_m(\zeta)}{c_2\Phi_h(\zeta)}$$

In this approach one has same style of fluxes computation as in model ALADIN.

2.

$$S(\zeta) = \begin{cases} 1 + \frac{1}{\alpha}(-\zeta)^{2/3} & for\,\frac{z}{L} < 0 (unstable) \\ 1 & for\,\frac{z}{L} \geq 0 (neutral\, and\, stable) \end{cases}$$

This is same approach as was used by Redelsperger, [5].

Now we can construct integral stratify function (Louis approach). It is apparent, that integral stratify functions will have different expression.

2.2 Integral stability function

Integration in surface layer is done with a surface-layer scaling parameter dependence, where interaction of $\Phi_m(\zeta)$ and $\Phi_h'(\zeta)$ is made. Integration of formulas (11) and (12) yield next expressions:

$$u_* = |\vec{V}|\kappa\frac{15}{4}\psi_m^{'-1} \qquad \theta_{v*} = \Delta\theta_v\frac{\kappa}{R}\psi_h^{'-1} \tag{16}$$

where formulas for integral stability functions have forms:

$$\psi_m'(z/z_0, Ri) = \int_0^z \frac{\Phi_m(\zeta)dz}{(z+z_0)} \qquad \psi_h'(z/z_0, Ri) = \int_0^z \frac{\Phi_h'(\zeta)dz}{(z+z_0)} \tag{17}$$

where $\Phi_h'(\zeta) = \frac{\Phi_m(\zeta)}{\varphi(\zeta)}$

From formulas (11, 15) one has relation for stationary TKE in the surface layer:

$$e_* = \alpha|\vec{V}|^2(\kappa G_e)^2 \quad \text{or in equivalent form:} \quad e_* = \frac{4}{15c_m c_\epsilon}|\vec{V}|^2(AG_e)^2 \tag{18}$$

$$\text{where} \quad A = \alpha^{3/2}c_\epsilon\kappa \quad \text{and} \quad G_e = \frac{1}{\int_0^z \frac{\Phi_m(\zeta)dz}{S(\zeta)^{1/2}(z+z_0)}}$$

For the plan to use TKE as prognostic value is necessary to use formula (8) for computation of momentum flux. In this formula, L_k needs exact formulation. Analogically in TKE prognostic equation it requires formulation for L_ϵ. For this reason we can write expressions with help of formulas (8, 10, 11, 14):

$$L_{km} = AG_k(Ri, z/z_0) \quad L_{kh} = AH_k(Ri, z/z_0) \tag{19}$$

$$L_{\epsilon m} = AG_\epsilon(Ri, z/z_0) \quad L_{\epsilon h} = AH_\epsilon(Ri, z/z_0) \tag{20}$$

where

$$H_k(Ri, z/z_0) = \frac{1}{\int_0^z \frac{\Phi_m(\zeta)S(\zeta)^{1/2}dz}{\varphi_i(\zeta)(z+z_0)}} \quad H_\epsilon(Ri, z/z_0) = \frac{1}{\int_0^z \frac{\Phi_m(\zeta))f(\zeta)dz}{S(\zeta)^{3/2}\varphi_i(\zeta)(z+z_0)}}$$

$$G_k(Ri, z/z_0) = \frac{1}{\int_0^z \frac{\Phi_m(\zeta)S(\zeta)^{1/2}dz}{(z+z_0)}} \quad G_\epsilon(Ri, z/z_0) = \frac{1}{\int_0^z \frac{\Phi_m(\zeta)f(\zeta)dz}{S(\zeta)^{3/2}(z+z_0)}}$$

For momentum and heat flux from (5) we have next formulas in surface layer:

$$\overline{u_i u_j} = \frac{2}{3}\delta_{ij}e - K_m|\vec{V}| \qquad K_m = \frac{4}{15c_m}e^{1/2}AG_k(Ri, z/z_0) \tag{21}$$

$$\overline{u_i\theta_v} = -K_h\Delta\bar{\theta_v} \qquad K_h = \frac{2}{3c}e^{1/2}AH_k(Ri, z/z_0) \tag{22}$$

From these derivation seems that for integral stability profiles we have to introduce differ mixing length (dissipative length) for momentum and for heat fluxes too.

Derivation of integral properties L_{km} and $L_{\epsilon m}$ were done for relation surface velocity to surface TKE.

2.3 Parameterization constants setting

From interactive computation of surface-layer scaling parameter (implicit relations in formula) we construct profiles of integral stability function for momentum, heat, TKE and eddy lengths L_k and L_ϵ for different parameterized constants setting. The setting of coefficients is very important for system "behavior". The origin is in present parameterized formulas. In this way we adjust the shape of stratify functions $f(Ri)$ and $\varphi_i(Ri)$.

In this article we present only one possible setting to have short overview.

$c_m = 3.115 \quad c_\epsilon = 0.831 \quad c = 5.639$

$c_0 = 1.2 \quad \kappa = 0.4 \quad \alpha = 3.75$

For this coefficients settings we investigate cases (1) and (2) from chapter (2.1).

In case (1) $S(\zeta) = 1$ we have similar computation of fluxes as we told above in model ALADIN. For this reason we examine integral stability functions. One can verify, for example, formula for momentum flux:

$$u_*^2 = \frac{4}{15c_m} e^{1/2} A G_k(Ri, z/z_0)|\vec{V}| \tag{23}$$

In this case $G_e = G_k$ and $\frac{4}{15c_m\kappa}(\alpha)^{1/2} = 1/A$. The momentum flux has same formulation as in model ALADIN now:

$$u_*^2 = \kappa^2 |\vec{V}|^2 G_k^2$$

For solving prognostic TKE (see bellow) this approach is used for beginning. Formula (23) is used for computation of momentum flux. When we use notation for ALADIN integral stability function as $G_k(Ri, z/z_0) = F_m(Ri, z/z_0)^{1/2}$, one can write:

$$u_*^2 = \kappa^2 |\vec{V}|^2 F_m \quad \rightarrow \quad u_*^2 = \frac{4}{15c_m} e^{1/2} A F_m^{1/2} |\vec{V}|$$

where for unstable case it has expression

$$F_m = 1 - \frac{2bRi}{1 + 2bd_m C_{nM}\sqrt{-Ri(1 + z/z_0)}}$$

where b is a constant for determining exchange coef. nearest neutrality, d_m is a constant for determining asymptotic unstable regime and C_{nM} is the drag coefficient in neutral air condition. For stable case it has expression

$$F_m = \frac{1}{1 + 2bRi(1 + dRi)^{-1/2}}$$

where d is a constant for determining asymptotic stable regime.

For heat flux one can write formula:

$$u_*\theta_{v*} = c_2 \kappa^2 |\vec{V}| \Delta\theta_v G_k H_k \quad \rightarrow \quad u_*\theta_{v*} = \frac{2}{3c} e^{1/2} A F_h^{1/2} \Delta\theta_v$$

To be completely compatible with current ALADIN computation we can do next modification:

$$u_*\theta_{v*} = \frac{2}{3c} e^{1/2} A \frac{F_h}{F_m^{1/2}} \Delta\theta_v$$

where F_h is an integral stability function for heat from model ALADIN (very similar to F_m, differences are in the constant setting). On the last we can write formula for surface stationary TKE:

$$e_* = \frac{4}{15c_m}|\vec{V}|^2 A^2 F_m$$

This method of derivation do not include different variation of TKE and surface velocity. It means that anisotropy properties of turbulence are less described. One can suppose from these results, without to do deep analyses, that $L_{km} = L_{\epsilon m}$. It is not true, because one needs to respect formulas (19, 20).

One know about possibility to improve expression for function $S(\zeta)$. There is an advantages to use stability function from ALADIN for comparison of results with current ALADIN computation, which gives no bed results for fluxes. We can take advantage of long time development of fluxes method computation in ALADIN. In one case it is advantage, but in other case it can be sometime disadvantage with over-elaborate of same estimation of processes.

This kind of derivation creates next results. Profile for TKE integral stability function (18), G_e^2 is plotted in figure (1a). The ratio of mixing and dissipation length one can see on the picture (1b). The dependence $\zeta(Ri)$ is plotted on picture (1d).

In the case (2) we have next results. Important is to know, that introducing of $S(\zeta)$ changes properties in unstable case only. When coefficients stay unaltered, there is no shift of the critical Richardson number in comparison to the previous case. For this reason we present the ratio of mixing and dissipation length for new coefficients setting $c_2 = 19.46$ and $c_3 = 13.63$, picture (1c).

2.4 Approximation of integral stability function by analytical functions

From previous chapter it seems that coefficients c_2 and c_3 have crucial role to the properties of integral stability function. Before to continue one can remember expressions for these coefficients by other parameterized constant:

$$c_2 = \frac{5c_m}{2c} \qquad c_3 = \frac{5c_m c_\epsilon}{cc_0}$$

There is a question if it is better to find approximative formula for L_{km} and $L_{\epsilon m}$ independently, or to find formula for L_{km} and for ratio of $L_{km}/L_{\epsilon m}$.

From analysis it is visible that different expression of $S(\zeta)$ function changes the slope of integral stability function in unstable case. Coefficient c_3, especially ratio of c_ϵ/c_0 emphasizes the influence of roughens parameter and has influence to the value of critical Richardson number. Certainly, coefficients c_2 and c_3 play together to the changes of critical Richardson number and roughness influence. Coefficient c_2 has main influence to the value in neutral condition, especially to the ratio of momentum and heat fluxes. These influences to the ratio of $L_{km}/L_{\epsilon m}$ are observed mainly in integral functions, because the original function with ζ dependence, as was shown, has not dependence to the dimensionless stability function (Businger empirical function).

Approximation of the ratio of $L_{km}/L_{\epsilon m}$ for case ($S(\zeta) = 1$) can be done with next function (similar to ALADIN stratification function for momentum):

$$M(Ri, z/z_0) = L_{km}/L_{\epsilon m} = \begin{cases} = & 1 - \frac{3bRi}{1+3bc\left[\frac{l_m}{z+z_0)}\right]^2[-Ri]^{1/2}} \quad Ri < 0 \\ = & \frac{1}{1+dRi(1+aRi)^{-1/2}} \quad Ri > 0 \end{cases} \tag{24}$$

where for this specific case: $b = 1$, $a = c = d = 1.1$ and l_m is a momentum mixing length from model ALADIN, for neutral air stratification. The l_m varies with hight.

Dependence L_{km} is adopted from current ALADIN scheme. Above surface layer we take again variance L_{km} from ALADIN model. Dissipation length $L_{\epsilon m}$ has variation with height in this moment: $L_{\epsilon m} = L_{km}/M(Ri, z/z_0)$.

3 Numerical integration

3.1 Different approaches of solving prognostic TKE

After a lot of algebraic manipulation with terms from 1Dimensional TKE prognostic equation (3), one can get forms, which is more appropriate for next computation. This equation has unknown square root of authentic TKE. We use notation $E = e^{1/2}$ in text bellow.

$$\frac{\partial E}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial z}(\rho W E) - \frac{fE}{\rho}\frac{\partial^2 E}{\partial z^2} - \frac{E}{\rho}\frac{\partial E}{\partial z}\frac{\partial f}{\partial z} - \frac{2f}{\rho}\left(\frac{\partial E}{\partial z}\right)^2 - \frac{c_\epsilon}{2L_{\epsilon m}}E^2 + F(z,t) \quad (25)$$

where $f = c_{2m}\rho(z)L_{km}$, ρ is a reference profile of density, W is a mean vertical velocity. Last term is a forced term $(F(z))$. We prescribe this term with using ALADIN integral stability functions for fluxes. One has expression for $F(z)$:

$$F(z,t) = \tfrac{2A}{15c_m}\left|\tfrac{\partial \vec{V}}{\partial z}\right|^2 \left(F_m^{1/2} - c_2 Ri \tfrac{F_h}{F_m^{1/2}}\right) \quad (26)$$

From (25) it seems that one need to solve a second order nonlinear partial differential equation (mixed type of parabolic and hyperbolic form). It is not possible find analytical solution. We have to find numerical appropriate scheme, which will be convergent to the "true" result. Considering the vertical mesh of ALADIN one must use finite element method for numerical solving of (25). There is many approaches to solve this equation.

In the next step we try to solve directly equation (25) in nonlinear mode. The purpose of solving of TKE equation after rewriting to discrete form is to find a zero of a system of (KLEV) nonlinear functions in (KLEV) variables by a modification of the Powell hybrid method. (KLEV) is a number of vertical level in numerical model.

The main principle of this method is to find approximative solution with interactive technique. The Jacobian, which is necessary for finding a gradient of solving vector, is calculated by a forward-difference approximation. For stability criteria we compute equation in implicit regime.

From results which were obtained by this method we have steel some difficulties. Problems can be involved by nonlinear instability. We try to suppress this problem with rewriting of differential formulas for derivatives. Filtering technique are used.

For this reason, to find the source of troubles, we try to find other method for solving of nonlinear TKE equation. In our case it can be useful more control the algorithm of computation, we try to prevent computation of approximation of Jacobian.

We try to take advantages from previous methods and to avoid observed problems. Next presented approach will use more strong simplification of (25). For this reason we use different formulation of TKE as was used for previous investigation. We use not square of TKE, but "full" TKE in TKE equation. One can write TKE equation now:

$$\frac{\partial e}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial z}(\rho W e) - \frac{1}{\rho}\frac{\partial}{\partial z}\left(f e^{1/2}\frac{\partial e}{\partial z}\right) - \frac{c_\epsilon}{L\epsilon}e^{3/2} + \sqrt{e}2F(z,t) \quad (27)$$

We turn back to the full TKE to prevent existence of TKE variable in denominator. In this equation we say that TKE is included in exchange coefficients in current time "0". This approach reduced previous equation to the form:

$$e^+ - e^0 = \Delta t \left[A + S^0 - \alpha \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho K_e^0 \frac{\partial e^+}{\partial z}\right) - \beta \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho K_e^0 \frac{\partial e^0}{\partial z}\right)\right] \tag{28}$$

where A is an advection, which is computed separately, α, β are coefficients for changing explicit regime to the implicit, S^0 is a forced term:

$$S^0 = -\frac{c_\epsilon}{L_{\epsilon m}}(e^0)^{3/2} + \sqrt{e^0}2F(z,t) \quad \text{and} \quad K_e^0 = c_{2m} L_{km} (e^0)^{1/2}$$

This formulation of prognostic TKE is very close to the formulation of diffusion equation. One can say, that it is a special linearization of (3).

TKE is computed in half model levels. To prevent problems with advection term we exclude this term from previous integration methods. Vertical advection is computed separately, it has close connection with dynamics of model at this time.

The stability criteria is changed from method to other methods. It has connection with choosing of approximation of derivatives, if the system is in implicit, explicit or semi-implicit form.

Approximative and common approach for estimation of stability can be done for all methods when we take in the aspect fact, that diffusion term is dominant in TKE equation.

For the last method one can write discrete approximation of (28) equation:

$$\begin{aligned}
e_j^+ - e_j^0 \quad = \quad & \Delta t S_j^0 + \\
& \alpha g c_{2m} \frac{\Delta t}{2(p_k - p_{k+1})}\left(g\rho_k (K_{j-1}''^0 + K_j''^0)\frac{(e_{j-1}^+ - e_j^+)}{\Phi_{j-1} - \Phi_j} - \right. \\
& \left. g\rho_{k+1}\,(K_{j+1}''^0 + K_j''^0)\frac{(e_j^+ - e_{j+1}^+)}{\Phi_j - \Phi_{j+1}} \right) + \\
& \beta g c_{2m} \frac{\Delta t}{2(p_k - p_{k+1})}\left(g\rho_k (K_{j-1}''^0 + K_j''^0)\frac{(e_{j-1}^0 - e_j^0)}{\Phi_{j-1} - \Phi_j} - \right. \\
& \left. g\rho_{k+1}\,(K_{j+1}''^0 + K_j''^0)\frac{(e_j^0 - e_{j+1}^0)}{\Phi_j - \Phi_{j+1}} \right)
\end{aligned} \tag{29}$$

where $K_j''^0 = (L_{km})_j e_j^{1/2}$ and index "k" marks full level.

Stability analysis of previous discrete equation gives next results. From knowledges seems that when $\alpha = 1$, scheme is full implicit for diffusion term. Term K'' and ρ_k are alternating with vertical coordinate. For this reason scheme should be only conditionally stable.

Many others methods for solving of (27) TKE equation were investigated. The results are not presented in this article. All methods gives approximately same results. More robust and resistance properties has nonlinear approach.

To have stable scheme for the discrete equations above, the time step have to be very short, about 10s at this time. These techniques work only in diagnostic-prognostic regime. It means that only one tendency is computed, in the next step initialization is done from diagnostic values and one tendency is computed again.

On the next, roots of TKE equation must be positive. This property in numerical equations is very difficult to realize. The results from solving nonlinear method (25) are in the next pictures (2, 3). On the pictures one can see vertical profiles of TKE for 7hours integration. These methods are more sensitive to the variation of mixing length.

Other method, which eliminate absence of clear prognostic regime, is method which has origin in (29) equation. This method with special average technique for diffusion term eliminates problems with numerical stability. For this method full prognostic regime can be used. Upper limit for time step is about 50s in this method. Some results and comparisons are presented in the pictures (2,3).

4 Conclusion

The solved problems are more complicated than to find stable scheme. The roots of our system should be positive, because it represents TKE or square of TKE. For this reason we expect a "balance" behavior. Mathematically we can formulate definition of our problem in following way. The inverse equation operator should has positive matrix elements and right side must be positive. On this time we have positive real roots with positive right (forcing) side (This definition is may be incomplete.). To have positive forcing we need to analyze buoyant and shear terms together with dissipation term. For the instable case there is no problem, the forced term is always positive.

For stable case situation is more complicated. In neutral case momentum and heat flux is differed by inverse Prandtl number $c_2 = 1/R$, where c_2 is a parameterized constant and R is a ratio of heat and momentum flux in neutral atmospheric condition. It means that drag coefficient for heat flux is biggest than drag coefficient for momentum flux for neutral air stratification. The contribution to the final forced term is more significant from buoyant part at this moment. It means that $F(z,t) < 0$, where $F(z,t)$ is a forced term which contains shear and buoyant production or consumption terms in TKE equation. In this moment dissipation term and source terms from time "0" has the important role. The sign should be turned with these terms. For these properties it is important to have restriction for critical Richardson number (In this article we use $Ri_c = 0.2$). It should not be "big" value, for reason to have no "big" negative value for $F(z,t)$. These features are significant in upper (model levels) atmosphere, where stable stratification of atmosphere is observed frequently.

In PBL layer or in the border troposphere and tropopause other problems create obstructions. In these layers we observe frequently strong turbulent mixing. It has influence to the mixing length. When the turbulence occurred in thickness layer we have also sharp fluctuation of mixing length. Increasing of mixing length increases exchange of coefficient dramatically. This feature has close connection with numerical stability of scheme and with feature of positive inverse matrix operator properties. It is the main problem for all methods.

From results which are not completely presented one can say, that the robust method is direct using of nonlinear equation. There exists limitation too, but it can be improved. The negative fluctuation effect of mixing length is increased with shallow convection. For this reason there is no possibility to solve prognostic TKE equation with these methods with using current shallow convection scheme. One way how to implement saturation impressed by turbulence is implementation of liquid water to the variable of potential temperature. Without shallow equation scheme one can compute the tendency of TKE.

The methods which do not use completely TKE equation as prognostic equation are comparable to current ALADIN scheme for computation of fluxes, it is not contain direct knowledge of "history" from previous time step. Difference is in nonlinear approach of computation and in fact that exchange coefficients enter to the diffusion equations for mean momentum and heat in the time "+".. For these properties we can solve TKE equation in "prognostic-diagnostic" regime with very short integration step (above 10s). To keep stability in appropriate range in all time step we do initialization TKE profile with stationary TKE and consecutive we compute tendency of TKE with described methods. This manner is good for scientific investigation, but not good for operative weather forecast. For longest time step in the case of rapid changes of mean variables in vertical direction, we obtain bellow or above this region negative value for TKE (effects of discrete mesh), what is not allowed. Without initialization with stationary TKE after few minutes model blow up.

To compare these previous methods of solving nonlinear equation with full prognostic method (averaging technique for part of diffusion operator) we accomplish some experiments. Results show some similar problems with previous methods. In the region with sharp varied vertical profile discontinuity are observed. Main difference is that full prognostic regime has a memory of turbulence. The biggest differences are observed in PBL layer in this case.

References

[1] Jean.F. Louis, A parametric model of vertical eddy fluxes in the atmosphere, *Boundary Layer Meteorology*, 17 (1979), pp. 187-202.

[2] G. Sommeria, Three-Dimensional Simulation of Turbulent Processes in an Undisturbed Trade Wind Boundary Layer, *Journal of the atmospheric sciences*,Vol. 33, (1975), pp. 216-241.

[3] J.A. Businger, J.C. Wyngaard, Y. Izumi, E.F. Bradley, Flux-Profile Relationship in the Atmospheric Surface Layer, *Journal of the atmospheric sciences*,Vol. 28, (1971), pp.181-189.

[4] J.F.Lois, M.Tiedke, J.F.Geleyn, A short history of the operational PBL-parameterization at ECMWF

[5] J.L. Redelsperger, F. Mahe, P. Carlotti, A simple and general subgrid model suitable both for surface layer and free-stream turbulence , *Flow, Turbulence, and Combustion*, Vol.200, No.66, pp.453-475, 2000.

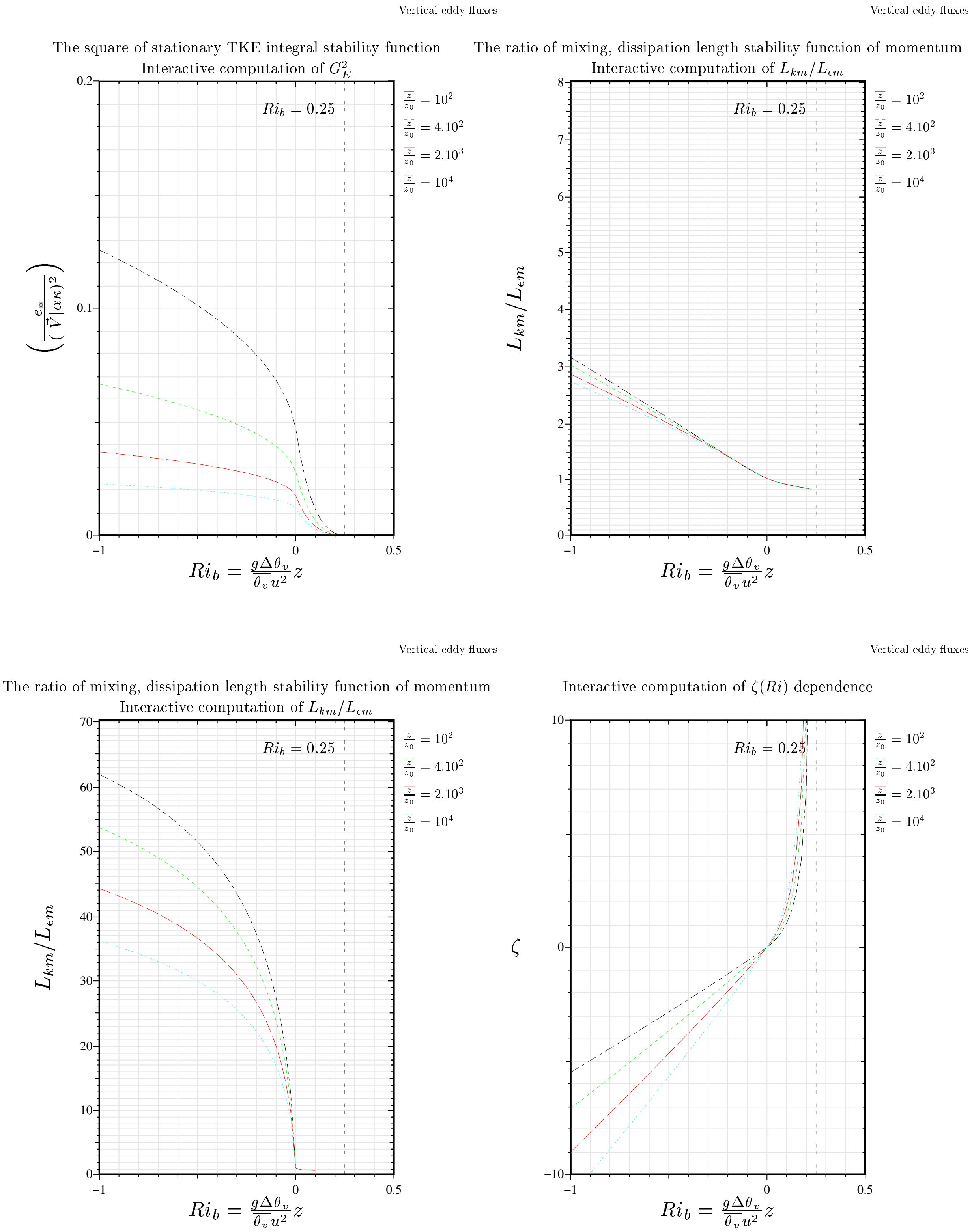

Figure 1: (a) The drag coefficients for turbulent energy flux with dependence on bulk Richardson number and the roughness length ($S(Ri = 1)$,$c_2 = 1/0.74$, $c_3 = 1.871$). (b) The ratio of mixing and dissipation length with dependence on bulk Richardson number and the roughness length. (c) The ratio of mixing and dissipation length with dependence on bulk Richardson number and the roughness length for different parameterized constant, $S(\zeta) = 1 + \frac{1}{\alpha}(-\zeta)^{2/3}$ for $\frac{z}{L} < 0$ and $S(\zeta) = 1$ for $\frac{z}{L} > 0$, $c_2 = 19.46$, $c_3 = 13.63$. (d) The dependence of the ζ on Ri and roughness parameter in the surface layer, constant settings are same as item (a,b).

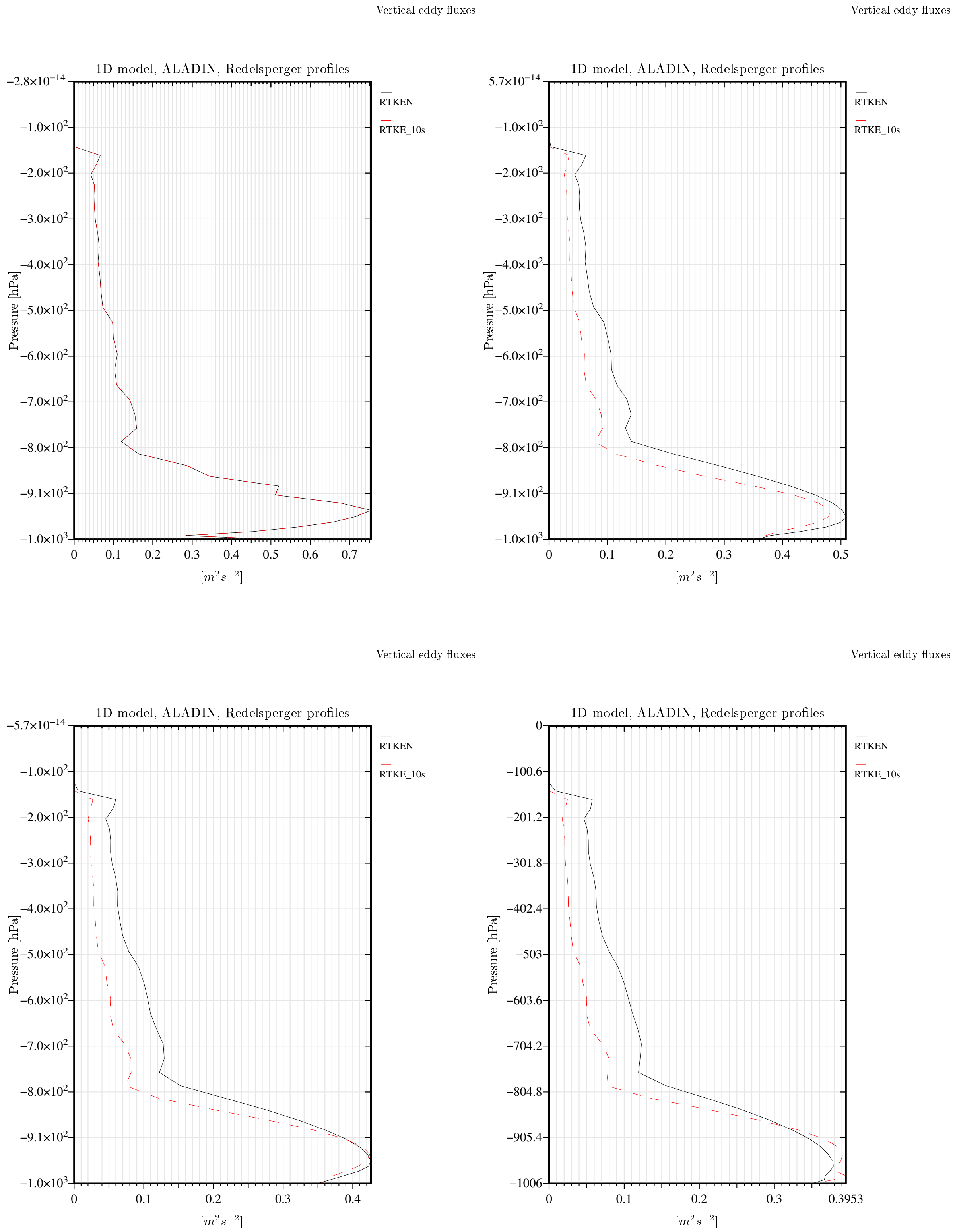

Figure 2: (a,b,c,d) The turbulent kinetic energy $[m^2s^{-2}]$ (RTKEN is computed by nonlinear diagnostic-prognostic method, RTKE is computed by simplified semi-implicit full prognostic method), for parameterized constants setting $c = 5.639$, $c_0 = 1.2$, $c_\epsilon = 0.831$, $c_m = 3.115$, $c_{2m} = 0.2$. For time step $\Delta t = 10s$, $USURICE = 5.0$. The left upper corner represents picture (a) for $t = 0h$, the right upper corner represents picture (b) for $t = 1h$, the lower left corner represents picture (c) for $t = 2h$ and the lower right corner represents picture (d) for $t = 3h$ of model integration.

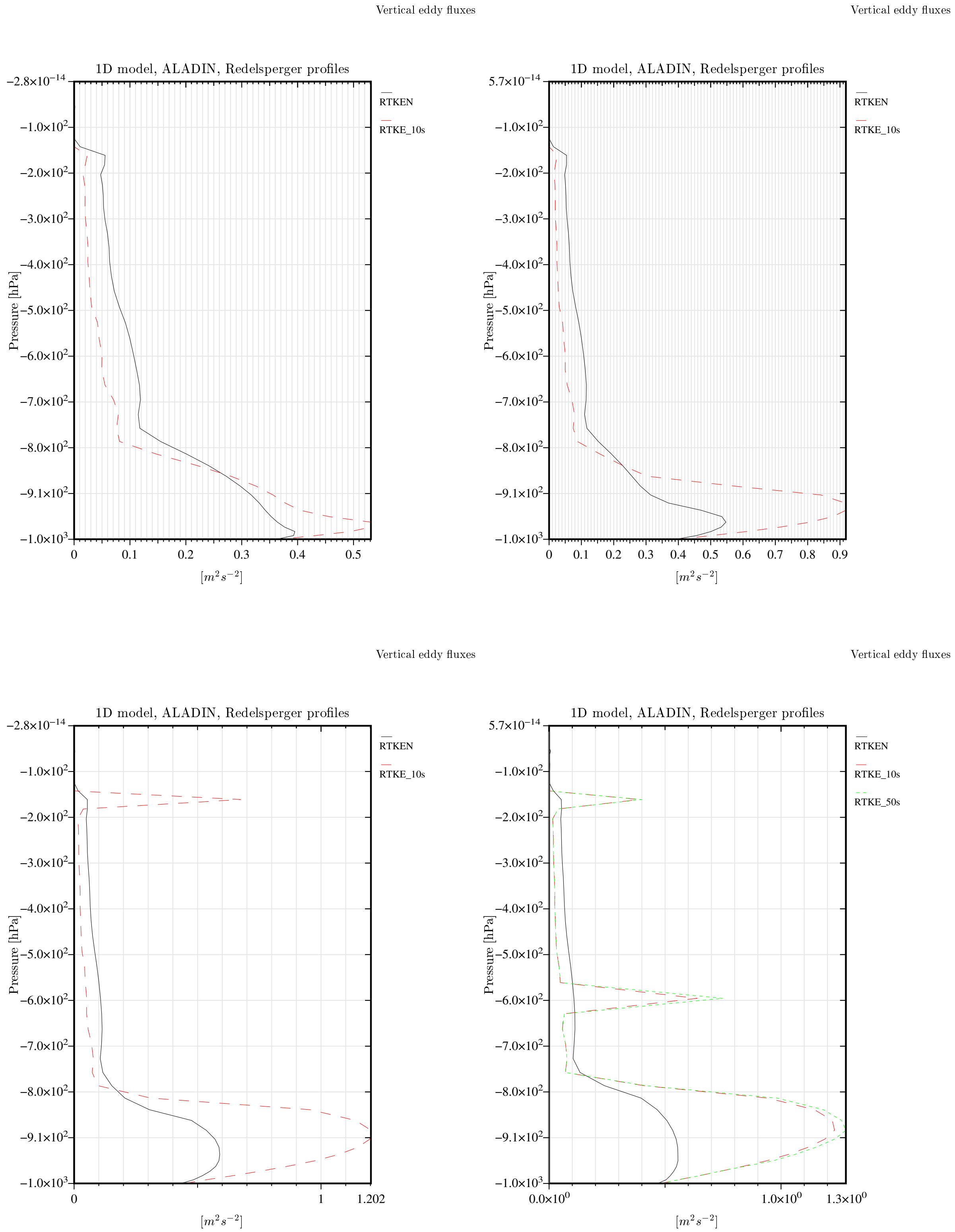

Figure 3: (a,b,c,d) The turbulent kinetic energy [m^2s^{-2}] (RTKEN is computed by nonlinear diagnostic-prognostic method, RTKE is computed by simplified semi-implicit full prognostic method), for parameterized constants setting $c = 5.639$, $c_0 = 1.2$, $c_\epsilon = 0.831$, $c_m = 3.115$, $c_{2m} = 0.2$. For time step $\Delta t = 10s, 50s, USURICE = 5.0$. The left upper corner represents picture (a) for $t = 4h$, the right upper corner represents picture (b) for $t = 5h$, the lower left corner represents picture (c) for $t = 6h$ and the lower right corner represents picture (d) for $t = 7h$ of model integration (RTKE for 50s integration time step is added).