Work document

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Summary of the changes between v1 and v2:

This version presents some modification after internal discussions following the comments of JF Geylein on the lack of anisotropy in the equations. The modifications are in **red**.

The anisotropy effects are added by removing one of the hypotheses of Redelsperger and Sommeria (1981) on the production terms in heat fluxes. This hypothesis was already pointed as questionable in the first version of the document (flagged as "Is it reasonable?"). Since it seems unreasonable, we propose now to remove it.

The new set of equations should now contains more of the physics that one wish to have. **This has to be confirmed during the working week. Additional modifications of the set of equation is of course possible if some terms are still missing**. One also need to confirm the form of the equation for the dissipation length/time.

Modification of the turbulence scheme to fit the TPE approach

Introduction

The objective is to modify the Cuxart et al 2000 turbulence scheme (used in ARPEGE, AROME and MesoNH) in stable layers. This scheme is formally identical to the one from Redelsperger et Soméria (1981), with in addition: The 1D version (used in operational models and in MesoNH at the mesoscale), turbulence of scalars, generalization to any orography and better turbulence treatment near the surface. The mixing length (Bougeault and Lacarrère 1989 is used in 1D turbulence, Deardorff mixing length is used in 3D turbulence, Redelsperger, Mahé et Carlotti 2001 is used in the SBL.

In addition, the shallow convection and dry convection is now treated by the Pergaud et al scheme (also named EDKF).

However, there are still some defaults:

1. **Mixing length**
   The Bougeault and Lacarrère (1989) mixing length does not work for neutral case.
2. **Anisotropy of the turbulence**
   The turbulence is supposed isotropic ($\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$) while this is false in convective BL ($\overline{w'^2}$ larger), in stable layers or near the surface (the opposite).
3. **Constants in the scheme**
   the critical richardson number is very small (0.17).

The objective is to correct these defaults in the scheme (except the neutral problem of the Bougeault and Lacarrere mixing length). To do that, one will incorporate some physics from the FMI team (Zilintinkevitch et al 2012) in the scheme. This requires to add a prognostic equation for TPE (Turbulent Potential Energy), or equivalently equations for the variance of potential (liquid) temperature, total water, and covariance of humidity and temperature.

In all this document, one supposes the mixing and dissipative length known! There may be some diagnostic or prognostic equations for those.

Contents

0.1 Closure Hypotheses of Reynolds equations

0.1.1 Basic Equations

$$\frac{\partial}{\partial t}(b_{ij}) +U_k \frac{\partial}{\partial x_k}(b_{ij}) = -\frac{\partial}{\partial x_k}\left(\overline{b'_{ij}u'_k}\right) -\frac{4}{3}eS_{ij} - \Sigma_{ij} - Z_{ij} +B_{ij} -\Pi_{ij}$$

$$\frac{\partial}{\partial t}(e) +U_k \frac{\partial}{\partial x_k}(e) = -\frac{\partial}{\partial x_k}\left(\overline{e'u'_k} + \overline{p'u'_k}\right) -\overline{u'_k u'_l}\frac{\partial U_k}{\partial x_l} +\beta_k\overline{u'_k\theta'_v} -\epsilon$$

$$\frac{\partial}{\partial t}(\overline{u'_i\theta'}) +U_k \frac{\partial}{\partial x_k}(\overline{u'_i\theta'}) = -\frac{\partial}{\partial x_k}\left(\overline{u'_k u'_i\theta'}\right) -\overline{u'_k\theta'}\frac{\partial U_i}{\partial x_k} - \overline{u'_i u'_k}\frac{\partial \Theta}{\partial x_k} +\beta_i\overline{\theta'\theta'_v} -\Pi_{i\theta}$$

$$\frac{\partial}{\partial t}(\overline{u'_i q'}) +U_k \frac{\partial}{\partial x_k}(\overline{u'_i q'}) = -\frac{\partial}{\partial x_k}\left(\overline{u'_k u'_i q'}\right) -\overline{u'_k q'}\frac{\partial U_i}{\partial x_k} - \overline{u'_i u'_k}\frac{\partial Q}{\partial x_k} +\beta_i\overline{q'\theta'_v} -\Pi_{iq}$$

$$\frac{\partial}{\partial t}(\overline{\theta'^2}) +U_k \frac{\partial}{\partial x_k}(\overline{\theta'^2}) = -\frac{\partial}{\partial x_k}\left(\overline{u'_k\theta'^2}\right) -2\overline{u'_k\theta'}\frac{\partial \Theta}{\partial x_k} -\epsilon_\theta$$

$$\frac{\partial}{\partial t}(\overline{q'^2}) +U_k \frac{\partial}{\partial x_k}(\overline{q'^2}) = -\frac{\partial}{\partial x_k}\left(\overline{u'_k q'^2}\right) -2\overline{u'_k q'}\frac{\partial Q}{\partial x_k} -\epsilon_q$$

$$\frac{\partial}{\partial t}(\overline{\theta' q'}) +U_k \frac{\partial}{\partial x_k}(\overline{\theta' q'}) = -\frac{\partial}{\partial x_k}\left(\overline{u'_k\theta' q'}\right) -\overline{u'_k\theta'}\frac{\partial Q}{\partial x_k} - \overline{u'_k q'}\frac{\partial \Theta}{\partial x_k} -\epsilon_{\theta q}$$

with:

$$b_{ij} = \overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}e$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\Sigma_{ij} = b_{ik}S_{kj} + S_{ik}b_{kj} - \frac{2}{3}\delta_{ij}b_{km}S_{mk}$$

$$Z_{ij} = R_{ik}b_{kj} - b_{ik}R_{kj}$$

$$R_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

$$B_{ij} = \beta_i\overline{u'_j\theta'_v} + \beta_j\overline{u'_i\theta'_v} - \frac{2}{3}\delta_{ij}\beta_k\overline{u'_k\theta'_v}$$

$$\beta_1 = \beta_2 = 0\ ;\ \beta_3 = \frac{g}{\theta}$$

0.1.2 Closure: dissipative terms

$$
\begin{aligned}
\epsilon &= C_\epsilon \frac{e^{\frac{3}{2}}}{L_\epsilon} \\
\epsilon_\theta &= 2C_{\epsilon_\theta} \frac{\sqrt{e}}{L_\epsilon} \overline{\theta'^2} \\
\epsilon_q &= 2C_{\epsilon_q} \frac{\sqrt{e}}{L\epsilon} \overline{q'^2} \\
\epsilon_{\theta q} &= 2C_{\epsilon_{\theta q}} \frac{\sqrt{e}}{L_\epsilon} \overline{\theta' q'}
\end{aligned}
$$

Values of the constants are given in the next paragraph. Those closures do not take into account the latest formulations for stable layers.

0.1.3 Closure: pressure terms

Note that the parameterizations of these terms presented here does not take into account latest Zilitinkevitch et al formulations yet. These should be incorporated if they differ from those presented.

$$
\begin{aligned}
\Pi_{ij} &= C_{pv} \frac{\sqrt{e}}{L}(b_{ij}) \quad -\tfrac{4}{3}\alpha_0 e S_{ij} - \alpha_1 \Sigma_{ij} - \alpha_2 Z_{ij} \quad +(1-\alpha_3) B_{ij} \\
\Pi_{i\theta} &= C_{p\theta} \frac{\sqrt{e}}{L}(\overline{u_i'\theta'}) \quad -\tfrac{3}{4}\alpha_4 \left(S_{ij} + \tilde{\alpha_4} R_{ij}\right) \overline{u_j'\theta'} \quad +\alpha_5 \beta_i \overline{\theta'\theta_v'} \\
\Pi_{iq} &= C_{pq} \frac{\sqrt{e}}{L}(\overline{u_i'q'}) \quad -\tfrac{3}{4}\alpha_4 \left(S_{ij} + \tilde{\alpha_4} R_{ij}\right) \overline{u_j'q'} \quad +\alpha_5 \beta_i \overline{q'\theta_v'}
\end{aligned}
$$

with:

	RS81 CBR00	SS89	KS92*/CD93* WLW96*	KC94	D'A98	N01	CCH02	???	comments
C_ϵ	0.7	0.845	0.845	*0.17*	*0.17*	*0.12*	$\left(\frac{2}{3K_o}\right)^{\frac{3}{2}} \pi$? (0.845)	0.845	K_o=1.6
C_{pv}	4	3.5	5.26/3.5/3.5	*0.51*	*0.68*	*0.4*	$\frac{5}{2}C_\epsilon$ (2.11)	2.11	
α_0	0.6	0.55	0.55	0.24	0.44	0.41	0.6	0.6	
α_1	-	-	-	-	-	-	0.984	-	
α_2	-	-	-	-	-	-	0.57	0.57	
α_3	0	0.45	0	0	1	0.35	0.5	0.5	
C_{ϵ_θ}	1.2	1.01	1.01	*0.2*	*0.09*	*0.09*	$\frac{1}{\sigma_0}C_\epsilon$ (1.01)	0.98	
$C_{p\theta}$	4	3.25	4.9/2.6/3.25	*0.64*	*0.91*	*0.71*	$\frac{5}{2}(1+\frac{1}{\sigma_0})C_\epsilon$ (4.65)	4.56	
α_4	-	-	-	0.93	-	0.266	0.285	-	
$\tilde{\alpha_4}$	-	-	-	1	-	1	$\frac{5}{3}$	-	
α_5	$\frac{1}{3}$	0	1	$\frac{1}{5}$	$\frac{1}{2}$	0.3	$\frac{1}{3}$	$\frac{1}{3}$	
R_{i_c}	0.139	0.2	0.2/0.27/0.2	0.21	0.76	0.28	0.96	?	
σ_0	0.4	0.42	0.42/0.33/0.42	0.8	1	1.35	0.82	0.86	$\sigma_0 = \frac{K_m(\zeta=0)}{K_h(\zeta=0)}$

The values come from:

RS81	Redelsperger et Someria (1981)
SS89	Schmidt et Schumann (1989)
KS92	Krettenauer et Schumann (1992)
CD93	Cuijpers et Duynkerke (1993)
KC94	Kantha et Clayson (1994)
WLW96	Wang, Large et McWilliams (1996)
D'A98	D'Alessio, Abdella et McFarlane (1998)
CBR00	Cuxart, Bougeault, Redelsperger (2000)
N01	Nakanishi (2001)
CCH02	Cheng, Canuto et Howard (2002)
???	possible choices (to iterate)

The schemes with an asterisk only use an exchange coefficient. The constants are found from those and from the dissipation constant, supposing α_0 from sources cited in the paper (SS89).

SS89 and KS92 do not use the same formalism for pressure terms (they have a term proportionnal to $e\frac{\partial\Theta}{\partial z}$ that nobody else has), but the simplifications made here allow to write them with the classical formalism and the constants written in the table.

Constants from the Yellor and Yamada dynasty are in italics.

When the constants are not written (-), this means that the term is neglected (both iin pressure terms and dynamical production). This means these constants value is 1.

The values of C_{pv}, $C_{p\theta}$, α_3 et α_4 in Cheng et al 2002, come from (Canuto and Dubovikov 1996a,b, 1997). Those for α_1 and α_2 come from Shih and Shabbir 1992, Canuto 1994.

The term with α_4 represents the interactions between different components of the heat flux. This term cancel in case of horizontal homogeneity (1D turbulence). So one choose to cancel this term : $\alpha_4 = 1$ et $\tilde{\alpha_4} = 1$.

Then:

$$\begin{aligned}
\Pi_{ij} &= C_{pv}\frac{\sqrt{e}}{L}(b_{ij}) \quad -\tfrac{4}{5}eS_{ij} - \Sigma_{ij} - \alpha_2 Z_{ij} \quad +(1-\alpha_3)B_{ij} \\
\Pi_{i\theta} &= C_{p\theta}\frac{\sqrt{e}}{L}(\overline{u_i'\theta'}) \quad +\tfrac{1}{3}\beta_i\overline{\theta'\theta_v'} \\
\Pi_{iq} &= C_{pq}\frac{\sqrt{e}}{L}(\overline{u_i'q'}) \quad +\tfrac{1}{3}\beta_i\overline{q'\theta_v'}
\end{aligned}$$

0.1.4 Tendency, advection by mean wind

All those terms are neglected for diagnosed equations. Prognostic equations are only for: TKE, variances of temperature, water, covariance of temperature and water.

Maybe the advection term could be kept only for TKE. This should be tested.

0.1.5 Temperature, humidity, Buoyancy notations

For sake of simplicity, in this document: θ stands for θ_l (the liquid potential temperature), q stands for q_t (the total water content) and buyancy is computed as:

$$\theta_v = E_\theta\theta + E_q q$$

The same holds for the flux:

$$\overline{w'\theta_v'} = E_\theta\overline{w'\theta'} + E_q\overline{w'q'}$$

0.1.6 System to solve

$$0 = -\frac{8}{15}e\frac{\partial U}{\partial x} \quad -2(1-\alpha_2)R_{12}\mathbf{b_{12}} \quad -2(1-\alpha_2)R_{13}\mathbf{b_{13}} \quad -\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w}'\theta'} \quad -\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w}'\mathbf{q}'} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{11}}$$

$$0 = -\frac{8}{15}e\frac{\partial V}{\partial y} \quad +2(1-\alpha_2)R_{12}\mathbf{b_{12}} \quad -2(1-\alpha_2)R_{23}\mathbf{b_{23}} \quad -\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w}'\theta'} \quad -\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w}'\mathbf{q}'} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{22}}$$

$$0 = -\frac{8}{15}e\frac{\partial W}{\partial z} \quad +2(1-\alpha_2)R_{13}\mathbf{b_{13}} \quad +2(1-\alpha_2)R_{23}\mathbf{b_{23}} \quad +\frac{4}{3}\alpha_3\beta E_\theta\overline{\mathbf{w}'\theta'} \quad +\frac{4}{3}\alpha_3\beta E_q\overline{\mathbf{w}'\mathbf{q}'} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{33}}$$

$$0 = -\frac{4}{15}e\frac{\partial U}{\partial y} - \frac{4}{15}e\frac{\partial V}{\partial x} \quad -(1-\alpha_2)R_{12}\mathbf{b_{22}} \quad -(1-\alpha_2)R_{13}\mathbf{b_{23}} \quad +(1-\alpha_2)R_{12}\mathbf{b_{11}} \quad +(1-\alpha_2)R_{32}\mathbf{b_{13}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{12}}$$

$$\begin{aligned} 0 = -\frac{4}{15}e\frac{\partial U}{\partial z} - \frac{4}{15}e\frac{\partial W}{\partial x} \quad & -(1-\alpha_2)R_{12}\mathbf{b_{23}} \quad -(1-\alpha_2)R_{13}\mathbf{b_{33}} \quad +(1-\alpha_2)R_{13}\mathbf{b_{11}} \quad +(1-\alpha_2)R_{23}\mathbf{b_{12}} \\ & +\alpha_3\beta E_\theta\overline{\mathbf{u}'\theta'} \quad +\alpha_3\beta E_q\overline{\mathbf{u}'\mathbf{q}'} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{13}} \end{aligned}$$

$$\begin{aligned} 0 = -\frac{4}{15}e\frac{\partial V}{\partial z} - \frac{4}{15}e\frac{\partial W}{\partial y} \quad & -(1-\alpha_2)R_{21}\mathbf{b_{13}} \quad -(1-\alpha_2)R_{23}\mathbf{b_{33}} \quad +(1-\alpha_2)R_{13}\mathbf{b_{12}} \quad +(1-\alpha_2)R_{23}\mathbf{b_{22}} \\ & +\alpha_3\beta E_\theta\overline{\mathbf{v}'\theta'} \quad +\alpha_3\beta E_q\overline{\mathbf{v}'\mathbf{q}'} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{23}} \end{aligned}$$

$$0 = -\frac{2}{3}e\frac{\partial \Theta}{\partial x} \quad -\mathbf{b_{11}}\frac{\partial \Theta}{\partial x} \quad -\mathbf{b_{12}}\frac{\partial \Theta}{\partial y} \quad -\mathbf{b_{13}}\frac{\partial \Theta}{\partial z} \quad -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{u}'\theta'}$$

$$0 = -\frac{2}{3}e\frac{\partial \Theta}{\partial y} \quad -\mathbf{b_{12}}\frac{\partial \Theta}{\partial x} \quad -\mathbf{b_{22}}\frac{\partial \Theta}{\partial y} \quad -\mathbf{b_{23}}\frac{\partial \Theta}{\partial z} \quad -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{v}'\theta'}$$

$$\begin{aligned} 0 = -\frac{2}{3}e\frac{\partial \Theta}{\partial z} \quad & -\mathbf{b_{13}}\frac{\partial \Theta}{\partial x} \quad -\mathbf{b_{23}}\frac{\partial \Theta}{\partial y} \quad -\mathbf{b_{33}}\frac{\partial \Theta}{\partial z} \quad +\frac{2}{3}\beta E_\theta\overline{\theta'^{\mathbf{2}}} \quad +\frac{2}{3}\beta E_q\overline{\theta'\mathbf{q}'} \\ & -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{w}'\theta'} \end{aligned}$$

$$0 = -\frac{2}{3}e\frac{\partial Q}{\partial x} \quad -\mathbf{b_{11}}\frac{\partial Q}{\partial x} \quad -\mathbf{b_{12}}\frac{\partial Q}{\partial y} \quad -\mathbf{b_{13}}\frac{\partial Q}{\partial z} \quad -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{u}'\mathbf{q}'}$$

$$0 = -\frac{2}{3}e\frac{\partial Q}{\partial y} \quad -\mathbf{b_{12}}\frac{\partial Q}{\partial x} \quad -\mathbf{b_{22}}\frac{\partial Q}{\partial y} \quad -\mathbf{b_{23}}\frac{\partial Q}{\partial z} \quad -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{v}'\mathbf{q}'}$$

$$\begin{aligned} 0 = -\frac{2}{3}e\frac{\partial Q}{\partial z} \quad & -\mathbf{b_{13}}\frac{\partial Q}{\partial x} \quad -\mathbf{b_{23}}\frac{\partial Q}{\partial y} \quad -\mathbf{b_{33}}\frac{\partial Q}{\partial z} \quad +\frac{2}{3}\beta E_\theta\overline{\theta'\mathbf{q}'} \quad +\frac{2}{3}\beta E_q\overline{\mathbf{q}'^{\mathbf{2}}} \\ & -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{w}'\mathbf{q}'} \end{aligned}$$

With four prognostic equations (that will not change further in the document), that do not cause any solving problem. In the other equations, e, $\overline{\theta'^2}$, $\overline{\theta' q'}$ and $\overline{q'^2}$ will be supposed known !

$$\frac{\partial}{\partial t}(\overline{\theta'^2}) \quad +U_k\frac{\partial}{\partial x_k}(\overline{\theta'^2}) \quad = \quad -\frac{\partial}{\partial x_k}\left(\overline{u'_k\theta'^2}\right) \quad -2\frac{\partial \Theta}{\partial x}\overline{\mathbf{u}'\theta'} \quad -2\frac{\partial \Theta}{\partial y}\overline{\mathbf{v}'\theta'} \quad -2\frac{\partial \Theta}{\partial z}\overline{\mathbf{w}'\theta'} \quad -2C_{\epsilon_\theta}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'^{\mathbf{2}}}$$

$$\frac{\partial}{\partial t}(\overline{q'^2}) \quad +U_k\frac{\partial}{\partial x_k}(\overline{q'^2}) \quad = \quad -\frac{\partial}{\partial x_k}\left(\overline{u'_k q'^2}\right) \quad -2\frac{\partial Q}{\partial x}\overline{\mathbf{u}'\mathbf{q}'} \quad -2\frac{\partial Q}{\partial y}\overline{\mathbf{v}'\mathbf{q}'} \quad -2\frac{\partial Q}{\partial z}\overline{\mathbf{w}'\mathbf{q}'} \quad -2C_{\epsilon_q}\frac{\sqrt{e}}{L_\epsilon}\overline{\mathbf{q}'^{\mathbf{2}}}$$

$$\begin{aligned} \frac{\partial}{\partial t}(\overline{\theta' q'}) \quad +U_k\frac{\partial}{\partial x_k}(\overline{\theta' q'}) \quad = \quad -\frac{\partial}{\partial x_k}\left(\overline{u'_k\theta' q'}\right) \quad & -\frac{\partial Q}{\partial x}\overline{\mathbf{u}'\theta'} \quad -\frac{\partial Q}{\partial y}\overline{\mathbf{v}'\theta'} \quad -\frac{\partial Q}{\partial z}\overline{\mathbf{w}'\theta'} \\ & -\frac{\partial \Theta}{\partial x}\overline{\mathbf{u}'\mathbf{q}'} \quad -\frac{\partial \Theta}{\partial y}\overline{\mathbf{v}'\mathbf{q}'} \quad -\frac{\partial \Theta}{\partial z}\overline{\mathbf{w}'\mathbf{q}'} \quad -2C_{\epsilon_{\theta q}}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'\mathbf{q}'} \end{aligned}$$

$$\frac{\partial}{\partial t}(e) \quad +U_k\frac{\partial}{\partial x_k}(e) \quad = \quad -\frac{\partial}{\partial x_k}\left(\overline{e'u'_k} + \overline{p'u'_k}\right) \quad -\overline{u'_k u'_l}\frac{\partial U_k}{\partial x_l} \quad +\beta_k\overline{u'_k\theta'_v} \quad -\epsilon$$

0.1.7 Additionnal hypothesis: dynamical production in heat flux equation

One follows here the hypothesis of Redelsperger et Soméria (1981): in equations for heat and mositure flux, the anisotropic terms are neglected when the isotropic terms are present.

This simplifies the resolution (separing somehow equations of heat and momentum).
Is it reasonable?

However, this hypothesis implies the removal of anisotropic forcing in the heat flux equation, that is crucial in stable layer. This would lead to replace $\overline{w'^2}$ by $\frac{2}{3}e$ in the dynamical production term. Therefore, the hypothesis does not seem reasonable and **we propose to reject it**.

However, we also propose to still **neglect the production terms involving wind fluxes**, keeping only the terms linked to wind variances (the most important one for stable layers being the term in $\overline{w'^2}$). All this will lead to a very slightly more complex equation to solve, but because the heat and water variance and covariances equations are prognostic (thanks to the TPE approach), this is easily solvable.

0.1.8 Additionnal hypothesis: pressure terms in wind variance equations

- The terms in $(1-\alpha_2)$ are neglected in the equations of wind flux (but not of variances).
- The terms due to horizontal heat fluxes are neglected in the equations of wind fluxes (but not of variances).

Those hypotheses imply that **the wind fluxes are primarily due to wind shear**.
Is it reasonable? What do you think of these hypotheses ?

0.1.9 Final system to solve (3D scheme)

$$0 = -\frac{8}{15}e\frac{\partial U}{\partial x} \quad -2(1-\alpha_2)R_{12}\mathbf{b_{12}} \quad -2(1-\alpha_2)R_{13}\mathbf{b_{13}} \quad -\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w}'\theta'} \quad -\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{11}}$$

$$0 = -\frac{8}{15}e\frac{\partial V}{\partial y} \quad +2(1-\alpha_2)R_{12}\mathbf{b_{12}} \quad -2(1-\alpha_2)R_{23}\mathbf{b_{23}} \quad -\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w}'\theta'} \quad -\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{22}}$$

$$0 = -\frac{8}{15}e\frac{\partial W}{\partial z} \quad +2(1-\alpha_2)R_{13}\mathbf{b_{13}} \quad +2(1-\alpha_2)R_{23}\mathbf{b_{23}} \quad +\frac{4}{3}\alpha_3\beta E_\theta\overline{\mathbf{w}'\theta'} \quad +\frac{4}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{33}}$$

$$0 = -\frac{4}{15}e\frac{\partial U}{\partial y} - \frac{4}{15}e\frac{\partial V}{\partial x} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{12}}$$

$$0 = -\frac{4}{15}e\frac{\partial U}{\partial z} - \frac{4}{15}e\frac{\partial W}{\partial x} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{13}}$$

$$0 = -\frac{4}{15}e\frac{\partial V}{\partial z} - \frac{4}{15}e\frac{\partial W}{\partial y} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{23}}$$

$$0 = -\overline{u'^2}\frac{\partial \Theta}{\partial x} \quad -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{u}'\theta'}$$

$$0 = -\overline{v'^2}\frac{\partial \Theta}{\partial y} \quad -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{v}'\theta'}$$

$$0 = -\overline{w'^2}\frac{\partial \Theta}{\partial z} \quad +\frac{2}{3}\beta E_\theta\overline{\theta'^2} \quad +\frac{2}{3}\beta E_q\overline{\theta'\mathbf{q}'} \quad -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{w}'\theta'}$$

$$0 = -\overline{u'^2}\frac{\partial Q}{\partial x} \quad -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{u'q'}}$$

$$0 = -\overline{v'^2}\frac{\partial Q}{\partial y} \quad -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{v'q'}}$$

$$0 = -\overline{w'^2}\frac{\partial Q}{\partial z} \quad +\frac{2}{3}\beta E_\theta\overline{\theta'\mathbf{q}'} \quad +\frac{2}{3}\beta E_q\overline{\mathbf{q'^2}} \quad -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{w'q'}}$$

0.1.10 Final system to solve (1D scheme)

$$0 = -(1-\alpha_2)\frac{\partial U}{\partial z}\mathbf{b_{13}} \quad -\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'} \quad -\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{11}}$$

$$0 = -(1-\alpha_2)\frac{\partial V}{\partial z}\mathbf{b_{23}} \quad -\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'} \quad -\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{22}}$$

$$0 = +(1-\alpha_2)\frac{\partial U}{\partial z}\mathbf{b_{13}} \quad +(1-\alpha_2)\frac{\partial V}{\partial z}\mathbf{b_{23}} \quad +\frac{4}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'} \quad +\frac{4}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}} \quad -C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{33}}$$

$$0 = \mathbf{b_{12}}$$

$$0 = -\frac{4}{15}e\frac{\partial U}{\partial z} - C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{13}}$$

$$0 = -\frac{4}{15}e\frac{\partial V}{\partial z} - C_{pv}\frac{\sqrt{e}}{L}\mathbf{b_{23}}$$

$$0 = \overline{\mathbf{u'}\theta'}$$

$$0 = \overline{\mathbf{v'}\theta'}$$

$$0 = -\overline{w'^2}\frac{\partial \Theta}{\partial z} \quad +\frac{2}{3}\beta E_\theta\overline{\theta'^{\mathbf{2}}} \quad +\frac{2}{3}\beta E_q\overline{\theta'\mathbf{q'}} \quad -C_{p\theta}\frac{\sqrt{e}}{L}\overline{\mathbf{w'}\theta'}$$

$$0 = \overline{\mathbf{u'q'}}$$

$$0 = \overline{\mathbf{v'q'}}$$

$$0 = -\overline{w'^2}\frac{\partial Q}{\partial z} \quad +\frac{2}{3}\beta E_\theta\overline{\theta'\mathbf{q'}} \quad +\frac{2}{3}\beta E_q\overline{\mathbf{q'^2}} \quad -C_{pq}\frac{\sqrt{e}}{L}\overline{\mathbf{w'q'}}$$

0.2 The 3D scheme

By subsitution, one have:

$$
\left\{
\begin{array}{lcl}
\overline{\mathbf{u'v'}} & = & -\frac{4}{15C_{pv}}L\sqrt{e}\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) \\
\overline{\mathbf{u'w'}} & = & -\frac{4}{15C_{pv}}L\sqrt{e}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right) \\
\overline{\mathbf{v'w'}} & = & -\frac{4}{15C_{pv}}L\sqrt{e}\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right) \\
\overline{\mathbf{u'^2}} & = & \frac{2}{3}e+\frac{L}{C_{pv}\sqrt{e}}\left\{-\frac{8}{15}e\frac{\partial U}{\partial x}+(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial U}{\partial y})^2+(\frac{\partial U}{\partial z})^2-(\frac{\partial V}{\partial x})^2-(\frac{\partial W}{\partial x})^2\right]-\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'}-\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}}\right\} \\
\overline{\mathbf{v'^2}} & = & \frac{2}{3}e+\frac{L}{C_{pv}\sqrt{e}}\left\{-\frac{8}{15}e\frac{\partial V}{\partial y}+(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial V}{\partial x})^2+(\frac{\partial V}{\partial z})^2-(\frac{\partial U}{\partial y})^2-(\frac{\partial W}{\partial y})^2\right]-\frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'}-\frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}}\right\} \\
\overline{\mathbf{w'^2}} & = & \frac{2}{3}e+\frac{L}{C_{pv}\sqrt{e}}\left\{-\frac{8}{15}e\frac{\partial W}{\partial z}+(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial W}{\partial x})^2+(\frac{\partial W}{\partial y})^2-(\frac{\partial U}{\partial z})^2-(\frac{\partial V}{\partial z})^2\right]+\frac{4}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'}+\frac{4}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}}\right\} \\
\overline{\mathbf{u'}\theta'} & = & -\frac{L}{C_{p\theta}\sqrt{e}}\overline{u'^2}\frac{\partial\Theta}{\partial x} \\
\overline{\mathbf{v'}\theta'} & = & -\frac{L}{C_{p\theta}\sqrt{e}}\overline{v'^2}\frac{\partial\Theta}{\partial y} \\
\overline{\mathbf{w'}\theta'} & = & -\frac{L}{C_{p\theta}\sqrt{e}}\left[\overline{w'^2}\frac{\partial\Theta}{\partial z}-\frac{2}{3}\beta E_\theta\overline{\theta'^2}-\frac{2}{3}\beta E_q\overline{\theta'\mathbf{q'}}\right] \\
\overline{\mathbf{u'q'}} & = & -\frac{L}{C_{p\theta}\sqrt{e}}\overline{u'^2}\frac{\partial Q}{\partial x} \\
\overline{\mathbf{v'q'}} & = & -\frac{L}{C_{p\theta}\sqrt{e}}\overline{v'^2}\frac{\partial Q}{\partial y} \\
\overline{\mathbf{w'q'}} & = & -\frac{L}{C_{pq}\sqrt{e}}\left[\overline{w'^2}\frac{\partial Q}{\partial z}-\frac{2}{3}\beta E_\theta\overline{\theta'\mathbf{q'}}-\frac{2}{3}\beta E_q\overline{\mathbf{q'^2}}\right]
\end{array}
\right.
$$

$$
\left\{
\begin{array}{llcl}
\frac{\partial}{\partial t}(\overline{\theta'^2}) & +\underbrace{U_k\frac{\partial}{\partial x_k}(\overline{\theta'^2})}_{\text{May be neglected?}} & = & -\frac{\partial}{\partial x_k}\left(\overline{\mathbf{u'_k}\theta'^2}\right) \quad -2\frac{\partial\Theta}{\partial x}\overline{\mathbf{u'}\theta'} \quad -2\frac{\partial\Theta}{\partial y}\overline{\mathbf{v'}\theta'} \quad -2\frac{\partial\Theta}{\partial z}\overline{\mathbf{w'}\theta'} \quad -2C_{\epsilon_\theta}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'^2} \\
\frac{\partial}{\partial t}(\overline{q'^2}) & +\underbrace{U_k\frac{\partial}{\partial x_k}(\overline{q'^2})}_{\text{May be neglected?}} & = & -\frac{\partial}{\partial x_k}\left(\overline{\mathbf{u'_k q'^2}}\right) \quad -2\frac{\partial Q}{\partial x}\overline{\mathbf{u'q'}} \quad -2\frac{\partial Q}{\partial y}\overline{\mathbf{v'q'}} \quad -2\frac{\partial Q}{\partial z}\overline{\mathbf{w'q'}} \quad -2C_{\epsilon_q}\frac{\sqrt{e}}{L_\epsilon}\overline{\mathbf{q'^2}} \\
\frac{\partial}{\partial t}(\overline{\theta'q'}) & +\underbrace{U_k\frac{\partial}{\partial x_k}(\overline{\theta'q'})}_{\text{May be neglected?}} & = & -\frac{\partial}{\partial x_k}\left(\overline{\mathbf{u'_k}\theta'\mathbf{q'}}\right) \quad -\frac{\partial Q}{\partial x}\overline{\mathbf{u'}\theta'} \quad -\frac{\partial Q}{\partial y}\overline{\mathbf{v'}\theta'} \quad -\frac{\partial Q}{\partial z}\overline{\mathbf{w'}\theta'} \\
 & & & \quad -\frac{\partial\Theta}{\partial x}\overline{\mathbf{u'q'}} \quad -\frac{\partial\Theta}{\partial y}\overline{\mathbf{v'q'}} \quad -\frac{\partial\Theta}{\partial z}\overline{\mathbf{w'q'}} \quad -2C_{\epsilon_{\theta q}}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'\mathbf{q'}} \\
\frac{\partial}{\partial t}(e) & +U_k\frac{\partial}{\partial x_k}(e) & = & -\frac{\partial}{\partial x_k}\left(\overline{e'\mathbf{u'_k}}+\overline{p'\mathbf{u'_k}}\right) \quad -\overline{\mathbf{u'_k}\mathbf{u'}_l}\frac{\partial U_k}{\partial x_l} \quad +\beta_k\overline{\mathbf{u'_k}\theta'_\mathbf{v}} \quad -\epsilon
\end{array}
\right.
$$

The only 'difficulty' remaining is the linear dependancy between vertical wind variance and heat and water fluxes. This can be solved by substituing the $\overline{w'\theta'}$ and $\overline{w'q'}$ expressions into the $\overline{w'^2}$ equation.

And that's all! Because all other terms on the right-hand-side are either known as mean variables or from the prognostic equations of variances (TKE, $\overline{\theta'^2}$, $\overline{\theta'q'}$, $\overline{q'^2}$). Note that equations for the heat and humidity vertical fluxes should be computed before the equations of variances.

0.3 Mesoscale model or GCM case: 1D scheme

$$\begin{cases} \overline{\mathbf{u'w'}} &= -\frac{4}{15C_{pv}}L\sqrt{e}\frac{\partial U}{\partial z} \\ \overline{\mathbf{v'w'}} &= -\frac{4}{15C_{pv}}L\sqrt{e}\frac{\partial V}{\partial z} \\ \overline{\mathbf{u'^2}} &= \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}(\frac{\partial U}{\partial z})^2 - \frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'} - \frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}}\right\} \\ \overline{\mathbf{v'^2}} &= \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}(\frac{\partial V}{\partial z})^2 - \frac{2}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'} - \frac{2}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}}\right\} \\ \overline{\mathbf{w'^2}} &= \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{-(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial U}{\partial z})^2 + (\frac{\partial V}{\partial z})^2\right] + \frac{4}{3}\alpha_3\beta E_\theta\overline{\mathbf{w'}\theta'} + \frac{4}{3}\alpha_3\beta E_q\overline{\mathbf{w'q'}}\right\} \\ \overline{\mathbf{w'}\theta'} &= -\frac{L}{C_{p\theta}\sqrt{e}}\left[\overline{w'^2}\frac{\partial\Theta}{\partial z} - \frac{2}{3}\beta E_\theta\overline{\theta'^{\mathbf{2}}} - \frac{2}{3}\beta E_q\overline{\theta'\mathbf{q'}}\right] \\ \overline{\mathbf{w'q'}} &= -\frac{L}{C_{pq}\sqrt{e}}\left[\overline{w'^2}\frac{\partial Q}{\partial z} - \frac{2}{3}\beta E_\theta\overline{\theta'\mathbf{q'}} - \frac{2}{3}\beta E_q\overline{\mathbf{q'^2}}\right] \end{cases}$$

With again the four prognostic equations:

$$\begin{cases} \frac{\partial}{\partial t}(\overline{\theta'^{\mathbf{2}}}) + \underbrace{U_k\frac{\partial}{\partial x_k}(\overline{\theta'^{\mathbf{2}}})}_{\text{May be neglected?}} = -\frac{\partial}{\partial z}\left(\overline{\mathbf{w'}\theta'^{\mathbf{2}}}\right) - 2\frac{\partial\Theta}{\partial z}\overline{\mathbf{w'}\theta'} - 2C_{\epsilon_\theta}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'^{\mathbf{2}}} \\ \frac{\partial}{\partial t}(\overline{q'^2}) + \underbrace{U_k\frac{\partial}{\partial x_k}(\overline{\mathbf{q'^2}})}_{\text{May be neglected?}} = -\frac{\partial}{\partial z}\left(\overline{\mathbf{w'q'^2}}\right) - 2\frac{\partial Q}{\partial z}\overline{\mathbf{w'q'}} - 2C_{\epsilon_q}\frac{\sqrt{e}}{L_\epsilon}\overline{\mathbf{q'^2}} \\ \frac{\partial}{\partial t}(\overline{\theta'\mathbf{q'}}) + \underbrace{U_k\frac{\partial}{\partial x_k}(\overline{\theta'\mathbf{q'}})}_{\text{May be neglected?}} = -\frac{\partial}{\partial z}\left(\overline{\mathbf{w'}\theta'\mathbf{q'}}\right) - \frac{\partial Q}{\partial z}\overline{\mathbf{w'}\theta'} \\ \qquad - \frac{\partial\Theta}{\partial z}\overline{\mathbf{w'q'}} - 2C_{\epsilon_{\theta q}}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'\mathbf{q'}} \\ \frac{\partial}{\partial t}(e) + U_k\frac{\partial}{\partial x_k}(e) = -\frac{\partial}{\partial z}\left(\overline{e'\mathbf{w'}} + \overline{p'\mathbf{w'}}\right) - \overline{\mathbf{u'w'}}\frac{\partial U}{\partial z} - \overline{\mathbf{v'w'}}\frac{\partial V}{\partial z} + \beta\overline{\mathbf{w'}\theta'_{\mathbf{v}}} - \epsilon \end{cases}$$

0.4 Particular case: dry 1D scheme

$$\begin{cases} \overline{\mathbf{u'w'}} &= -\frac{4}{15C_{pv}} L\sqrt{e}\frac{\partial U}{\partial z} \\ \overline{\mathbf{v'w'}} &= -\frac{4}{15C_{pv}} L\sqrt{e}\frac{\partial V}{\partial z} \\ \overline{\mathbf{u'^2}} &= \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}(\frac{\partial U}{\partial z})^2 - \frac{2}{3}\alpha_3\beta\overline{\mathbf{w'}\theta'}\right\} \\ \overline{\mathbf{v'^2}} &= \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}(\frac{\partial V}{\partial z})^2 - \frac{2}{3}\alpha_3\beta\overline{\mathbf{w'}\theta'}\right\} \\ \overline{\mathbf{w'^2}} &= \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{-(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial U}{\partial z})^2 + (\frac{\partial V}{\partial z})^2\right] + \frac{4}{3}\alpha_3\beta\overline{\mathbf{w'}\theta'}\right\} \\ \overline{\mathbf{w'}\theta'} &= -\frac{L}{C_{p\theta}\sqrt{e}}\left[\overline{w'^2}\frac{\partial\Theta}{\partial z} - \frac{2}{3}\beta\overline{\theta'^2}\right] \end{cases}$$

The equation for the vertical wind variance can be written as:

$$\overline{\mathbf{w'^2}} = \frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{-(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial U}{\partial z})^2 + (\frac{\partial V}{\partial z})^2\right] - \frac{4}{3}\alpha_3\beta\frac{L}{C_{p\theta}\sqrt{e}}\left[\overline{w'^2}\frac{\partial\Theta}{\partial z} - \frac{2}{3}\beta\overline{\theta'^2}\right]\right\}$$

i.e.

$$\overline{\mathbf{w'^2}} = \frac{1}{1+\frac{\frac{4}{3}\alpha_3\beta L^2}{C_{pv}C_{p\theta}e}\frac{\partial\Theta}{\partial z}}\left(\frac{2}{3}e + \frac{L}{C_{pv}\sqrt{e}}\left\{-(1-\alpha_2)\frac{4L\sqrt{e}}{15C_{pv}}\left[(\frac{\partial U}{\partial z})^2 + (\frac{\partial V}{\partial z})^2\right] + \frac{8}{9}\alpha_3\beta^2\frac{L}{C_{p\theta}\sqrt{e}}\overline{\theta'^2}\right\}\right)$$

This equation is solved first, then the heat flux equation and finally the horizontal wind variances (but those are only needed for diagnostic purposes in a 1D vertical turbulence scheme).

The two prognostic equations for heat variance and TKE are:

$$\begin{cases} \frac{\partial}{\partial t}(\overline{\theta'^2}) + \underbrace{U_k\frac{\partial}{\partial x_k}(\overline{\theta'^2})}_{\text{May be neglected?}} &= -\frac{\partial}{\partial z}\left(\overline{\mathbf{w'}\theta'^2}\right) \quad -2\frac{\partial\Theta}{\partial z}\overline{\mathbf{w'}\theta'} \quad -2C_{\epsilon_\theta}\frac{\sqrt{e}}{L_\epsilon}\overline{\theta'^2} \\ \frac{\partial}{\partial t}(e) + U_k\frac{\partial}{\partial x_k}(e) &= -\frac{\partial}{\partial z}\left(\overline{e'\mathbf{w'}} + \overline{p'\mathbf{w'}}\right) \quad -\overline{\mathbf{u'w'}}\frac{\partial U}{\partial z} - \overline{\mathbf{v'w'}}\frac{\partial V}{\partial z} \quad +\beta\overline{\mathbf{w'}\theta'} \quad -\epsilon \end{cases}$$

Note that:

$$TPE = \frac{1}{2}\frac{\beta}{\frac{\partial\Theta}{\partial z}}\overline{\theta'^2}$$

Then the buoyancy production term vanishes, as expected, when one considers the total turbulent energy ($TTE = e + TPE$).