Energy Estimates and Weak Boundary Procedures for LAM

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Outline

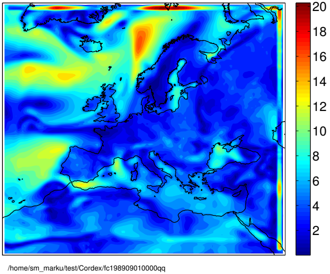
Introduction/Motivation for this work

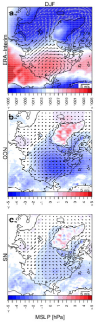
- Boundary conditions are set using *ad hoc* intuition (Davies relaxation)
- Different effects in e.g. Arctic and Europe
- "The apparent success of spectral nudging for one-way nesting is at least partly an artifact of very bad procedures for windowing and blending"[1]

[1] John P. Boyd "Limited-Area Fourier Spectral Models and Data Analysis Schemes: Windows, Fourier Extension, Davies Relaxation and All That", Mon. Weat. Rev. Vol. 133 2005

Motivating example

- Europe has "standing waves"
- Similar problems over Arctic (mitigated with Spectral Nudging)





- Analyze boundary procedures!

Model problem

Simplification for purpose of illustration, without loosing generality!

- $\vec{u}_t + \sum_{i=1}^{3} \nabla \vec{F}_i(\vec{u}) = 0$ N-S/Euler/primitive equations

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- $u_t + u_x = 0$

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(Energy point of view)

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- $\frac{d}{dt}\|u\|^2 = \underbrace{(u(0,t))^2}_{Faster \;\; than \;\; exponential \;\; growth} - \underbrace{(u(1,t))^2}_{Decay}$
- We must set conditions to bound the energy for $u(0, t)$ with data: $g(t) < C$ for some constant $C \in \mathcal{R}$

Discrete Boundary procedures

$$\text{CKD} \; : \; U_t + P^{-1}QU = 0$$
$$U(\cdot, t) = (I - W)U(\cdot, t) + WG(t) \qquad W = diag(tanh)$$

- G is data!
- $U = (U_0, U_1, \ldots U_N)^T$ is the solution.
- CKD = Classic Kållberg-Davies Relaxation

Discrete Boundary procedures

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Energy estimate for Davies-relaxation

Strong Davies relaxation

$$U_t + P^{-1}QU = 0$$

$$U = U + W_{tanh}(G - U)$$

- No energy estimate! (For stability(?) proof use GKS theory *DIFFICULT*!)

Weak Davies relaxation

$$U_t + P^{-1}QU = \tau P^{-1}W_{tanh}(G - U)$$

- Discrete Energy Method to prove stability
- Multiply with $U^T P$
- Use summation-by-parts property of difference operator

- In fact there are counterexamples shoving instability for strong methods.[2]

[2] "High Order Difference Methods for Time Dependent PDE", Bertil Gustafsson ISBN 978-3-540-74992-9, Springer Verlag 2008

Energy estimate for Davies-relaxation cont.

$$\frac{d}{dt}\|U\|_P^2 = U_1^2(1-2w_1) + 2w_1 U_1 G_1 - U_N^2(1+2w_N) + 2w_N U_N G_N - \sum_{i=2}^{N-1}(2w_i U_i^2 - 2w_i U_i G_i) \quad (1)$$

Since $w_1 \geq \frac{1}{2}$ and $w_i \geq 0 \therefore$ Davies Relaxation is *proven* energy-stable!

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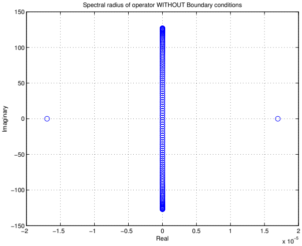
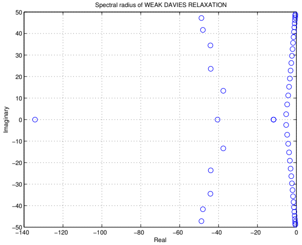
- Stable, *BUT* imposing conditions where not needed!
- Solution quality is not affected if $\|U_i - G_i\|$ is "small".
- Usually we impose time-interpolated G i.e. $G_i(t) = \pi_{6h} G_i(t_n)$

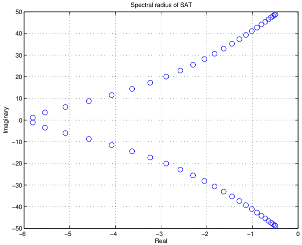
Spectral Radius of operators part I



(a) Operator without boundary conditions



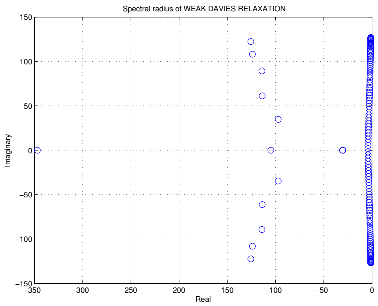
(b) WKD



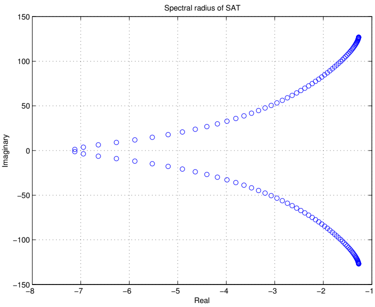
(c) SAT

Spectral Radius of operators part II

Increasing resolution by a factor of 5:



(d) WKD



(e) SAT

$\rho(WKD) \approx 350, \quad \rho(SAT) \approx 140$

Computational results

$$u_t + u_x = 0$$

$$u(0,t) = G(0,t), \quad x \in [0,1]$$

$$\text{Exact solution is } u(x,t) = sin(2\pi(x - t - \frac{1}{2})) = G(x,t)$$

We use the exact solution as initial data, i.e. assume perfect assimilation

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- Use exact $G(\cdot, t)$ as boundary data (show movie)

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- πG is constant interpolation in time (closest in time) (show movie)

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- Mismatching data on outflow boundary

Conclusion

- Davies relaxation in its weak form is a penalty method
- WKD is proven stable and yields similar results with CKD
- CKD and WKD impose data on all boundaries, even when not needed.
- SAT is proven stable, imposes data only where needed!
- The WKD operator is much more stiff than SAT by a factor of $\approx$20!
- When data is not close to the solution on outflow boundaries (and WKD) yields unsatisfactory results $\Rightarrow$ *horizontal diffusion mitigates this problem*
- The excessive diffusion can be the reason for the poor results over the Arctic
- Non-matching data causes a "standing wave" on the outflow boundary with WKD and CKD
- SAT yields similar results for exact and almost matching data (time interpolation error is visible), but outperforms CKD and WKD for non-matching data.

Conclusion II

- Theory for penalty-based boundary conditions is considered mature and ready to be used for NWP and climate
- Results are already extended to non-linear multidimensional systems, but for purpose of illustration a model problem was shown here.

Conclusion II

- THANK YOU FOR LISTENING!