Wavelet Representation of Background Error Covariance

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Contents

Introduction: diagonalising B

- In 3D-Var we want to minimise the cost function

$$J = J_b + J_o = \frac{1}{2}(\mathbf{x}-\mathbf{x}^b)^*\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}^b) + \frac{1}{2}(\mathbf{H}\mathbf{x}-\mathbf{y})^*\mathbf{R}^{-1}(\mathbf{H}\mathbf{x}-\mathbf{y}),$$

- If we represent $\mathbf{B}$ in grid space, the diagonal represents the error variance at every grid point.
- Diagonalising $\mathbf{B}$ in Fourier space ($\mathbf{B} = \mathbf{F}^*\mathbf{B}_f\mathbf{F}$): homogeneous
  (mean structure function at every location)
- Better: diagonalising the correlation matrix
  (mean correlation function, local variance)

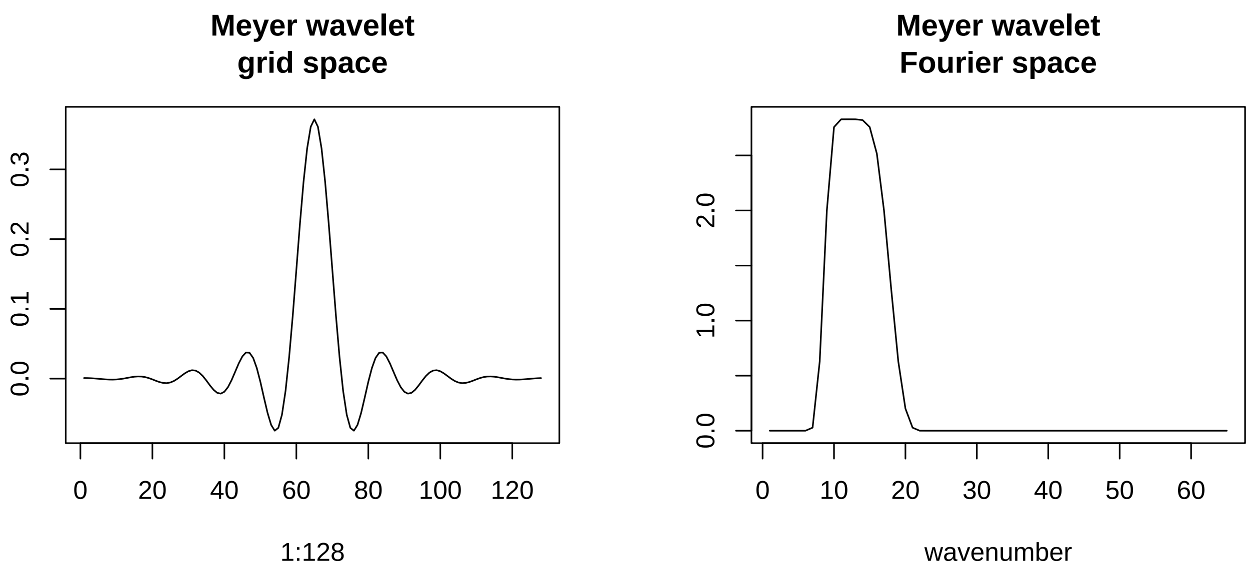
$$\mathbf{B} = \mathbf{D}^{1/2}\mathbf{F}^*\mathbf{C}_f\mathbf{F}\mathbf{D}^{1/2}$$

Introduction: diagonalising $\mathbf{B}$

- Can we simplify $\mathbf{B}$ in another way?
- We must still be able to calculate $\mathbf{B} = UU^*$
- Take account of local difference in variance and structure functions.

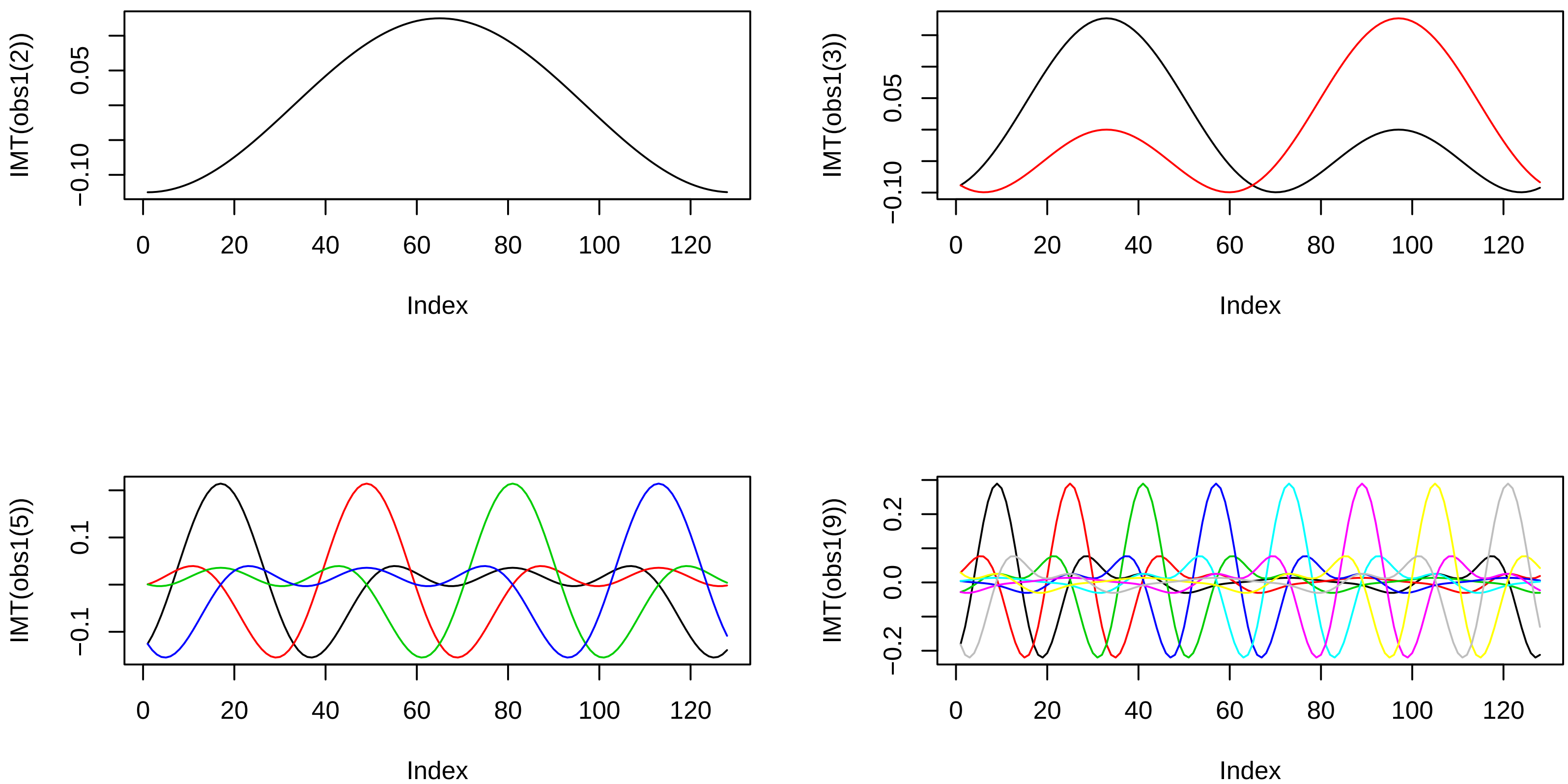
Wavelets

- Orthogonal Discrete Wavelet Transform
- Somewhere inbetween grid point and Fourier representation:
- Basis functions are localised both in grid and Fourier space $\longrightarrow$ Always some compromise!



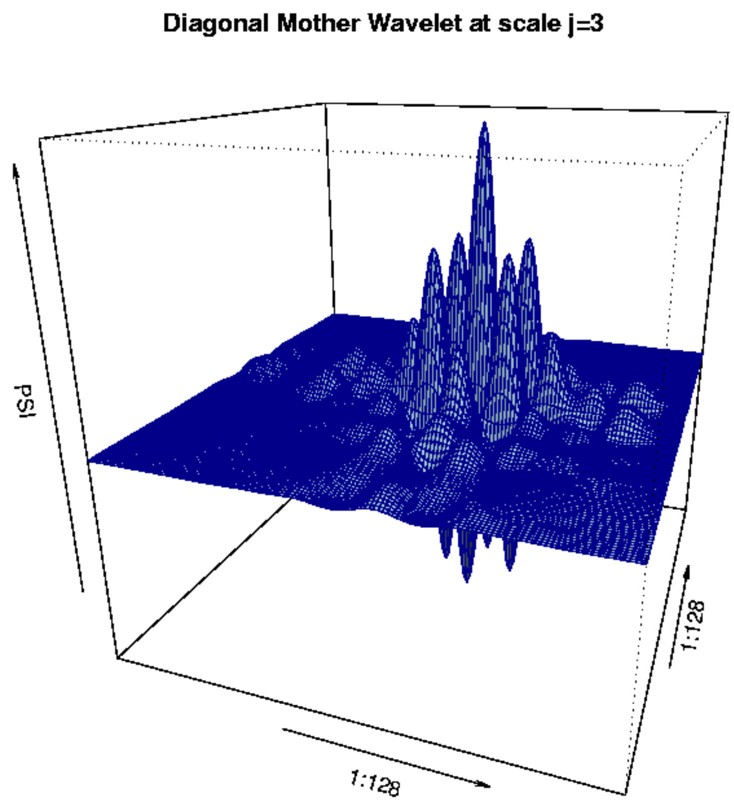
Wavelets

- These wavelets are repeated at different scales (usually powers of 2) and locations to form an orthogonal basis.



Example of a 2d wavelet

- Example of a 2d wavelet:



The 3 Bases Approach

- Use every basis for its strongest points:
  - Grid space: strictly local (variance)
  - Fourier space: average correlation function
  - Wavelet space: local differences from average

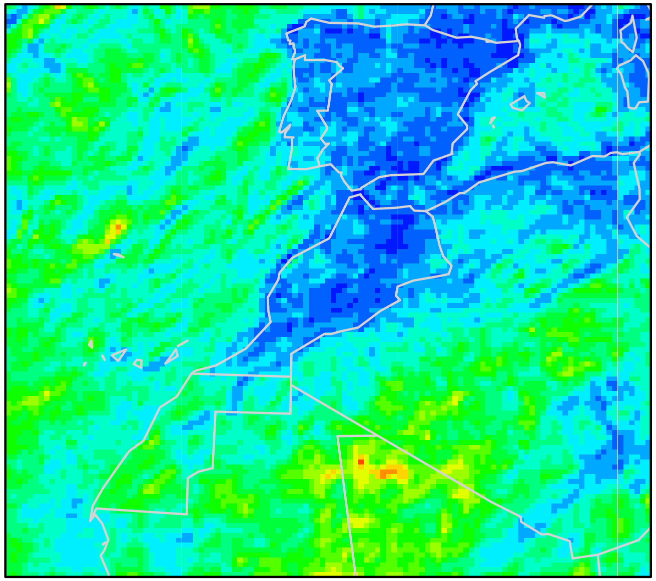
$$
\begin{aligned}
B &= D_g^* F^* D_f^* (F^{-1})^* W^* B_w W F^{-1} D_f F D_g, \\
B_w &= d \left\{ W F^{-1} D_f^{-1} F D_g^{-1} \overline{T^* T} D_g^{-1} F^{-1} D_f^{-1} F W^{-1} \right\}.
\end{aligned}
$$

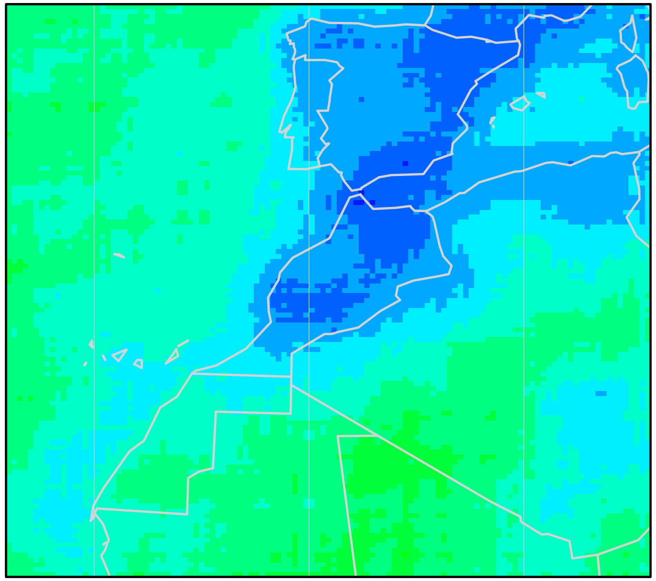
- In fact this generalises the spectral approach ($B = F^* B_f F$):

$$
B_f = D_f^* D_f \rightarrow D_f^* (F^{-1})^* W^* B_w W F^{-1} D_f
$$

Lengthscale

- Local lengthscales at surface:





Anisotropy

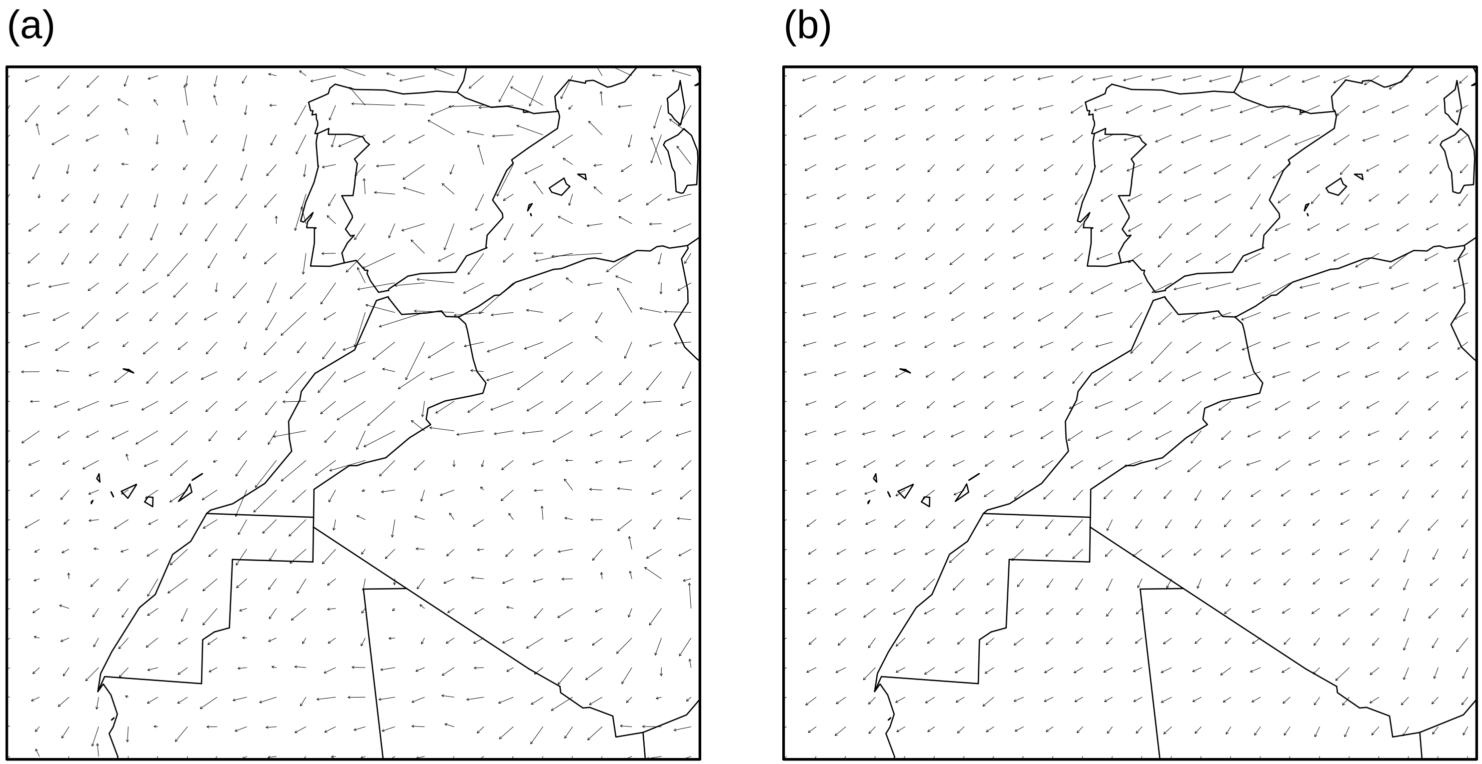


Figure 1: Local anisotropy axes at model level 31.

Anisotropy

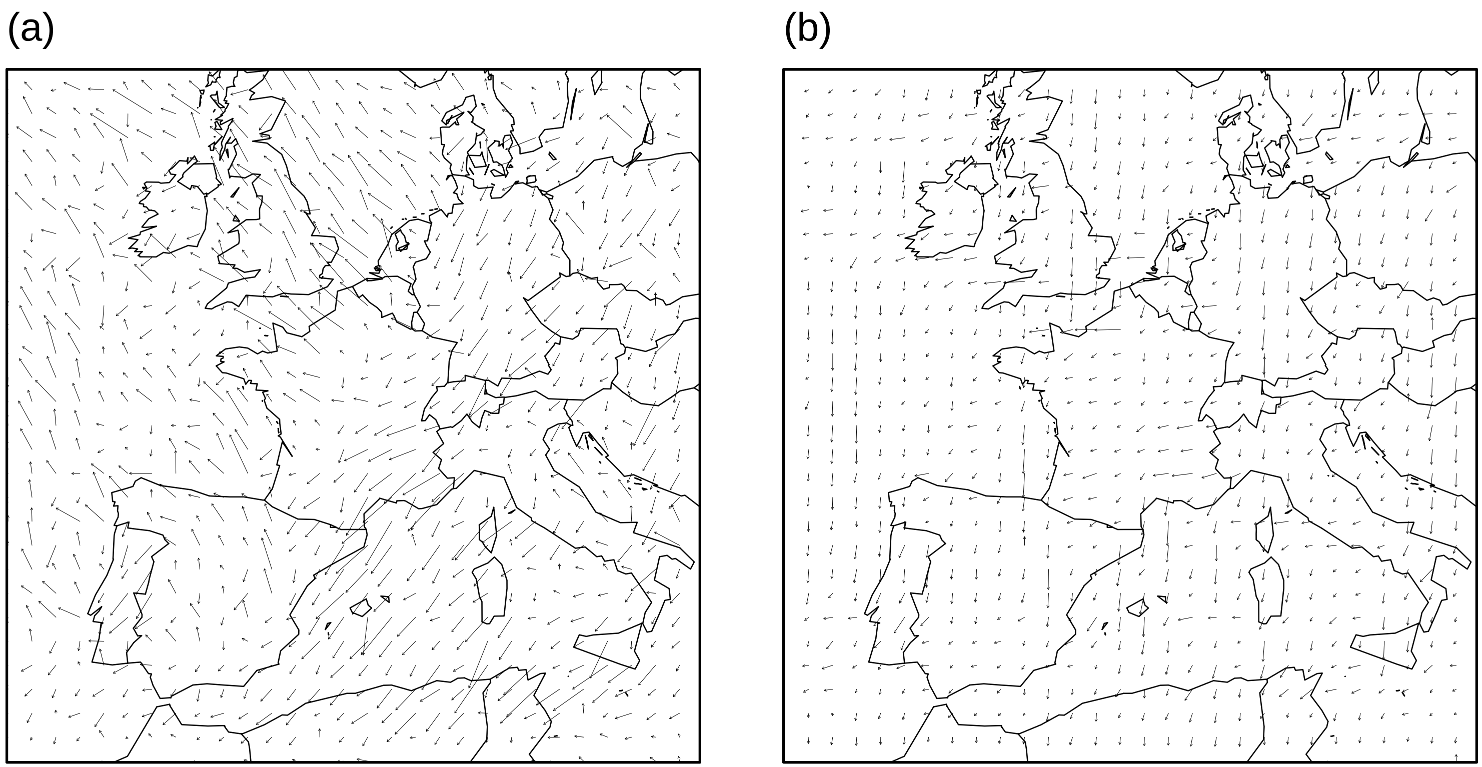
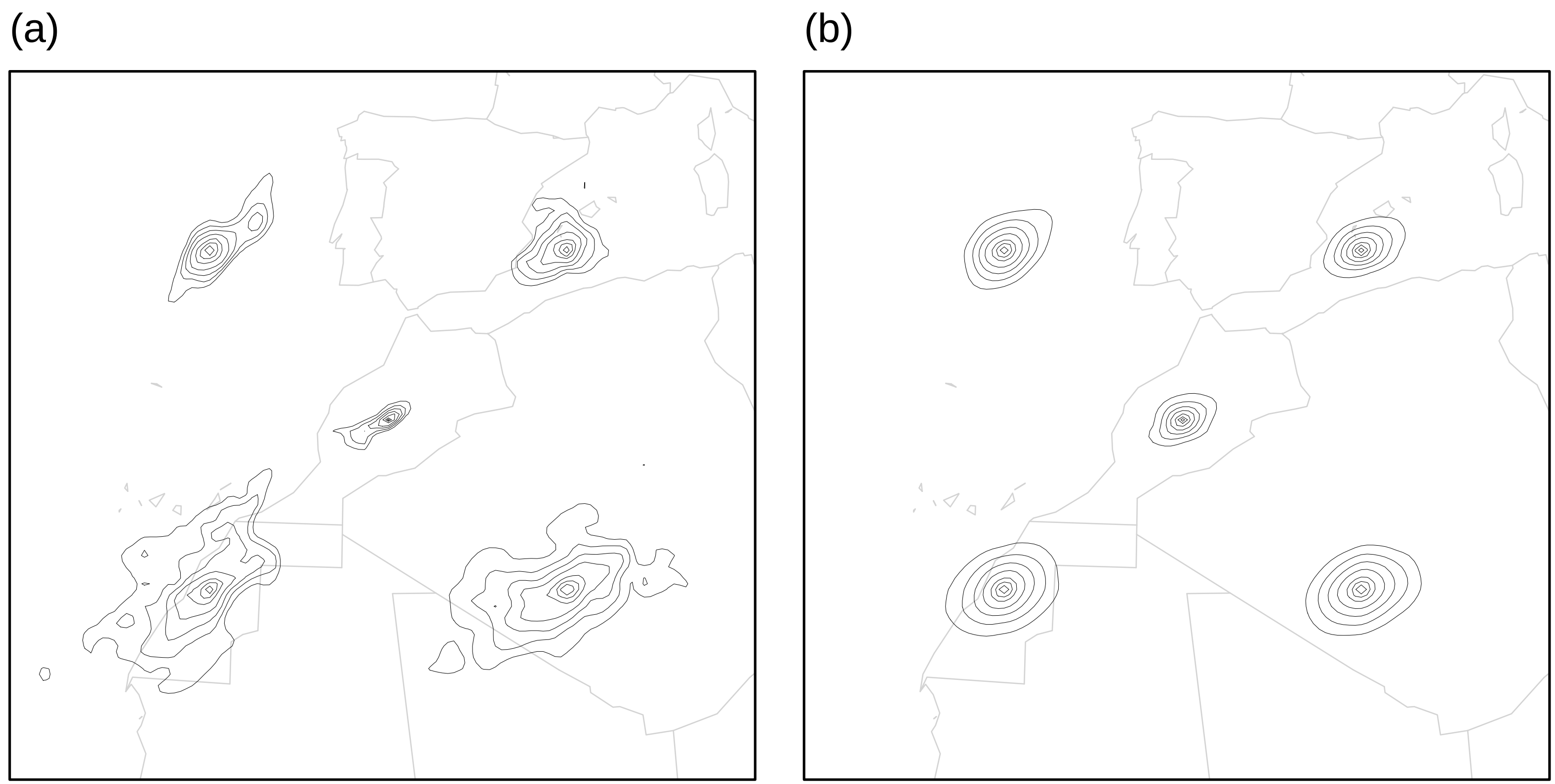


Figure 2: Local anisotropy axes at model level 41.

→ problem with diagonal (NW) directions...

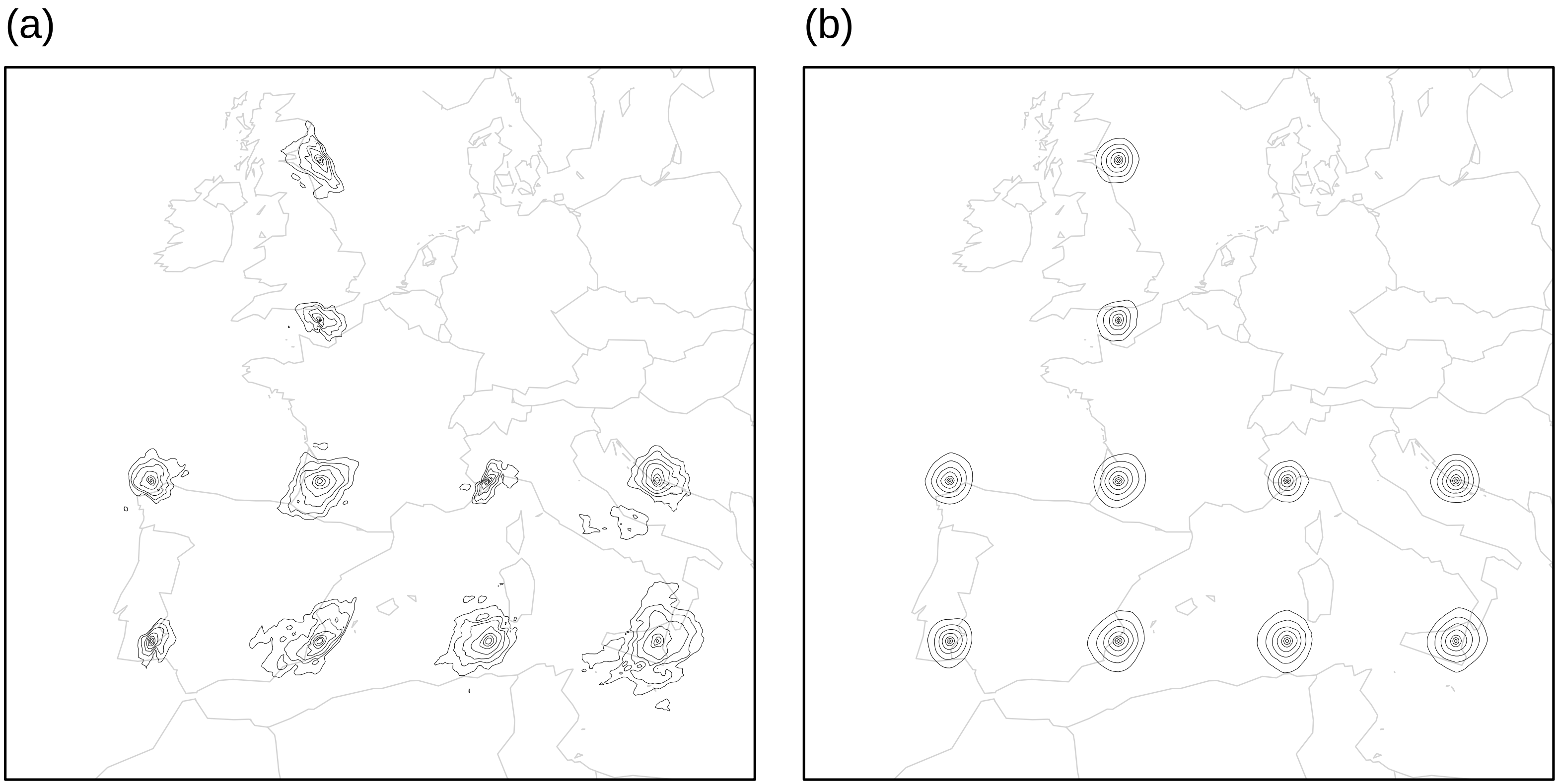
Correlation functions

- Level 31:



Correlation functions

- ALADIN/France lev 41:

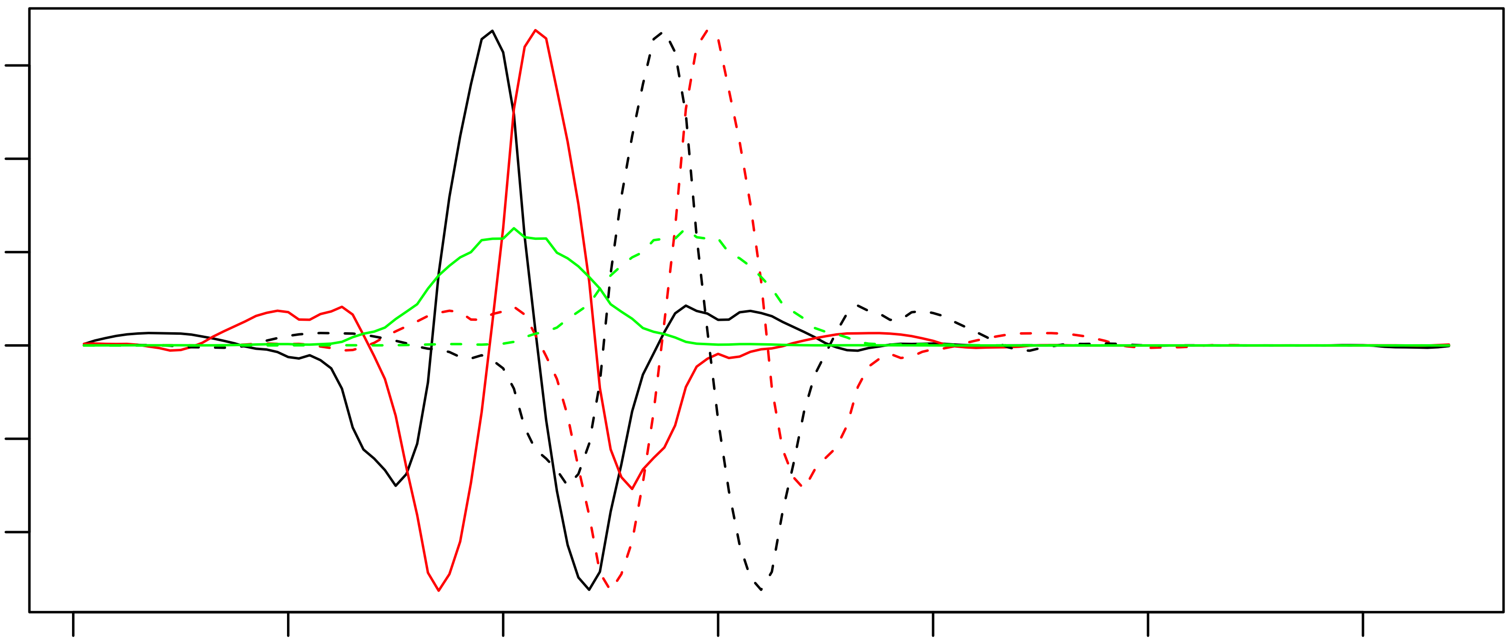


Halfway the presentation - status?

- Lengthscales are quite well captured.
- The anisotropy on ALADIN/France is not represented:
  - Our 2D wavelets wavelets are too much centered around X and Y axis.
  - The average anistropy is small (different regions)

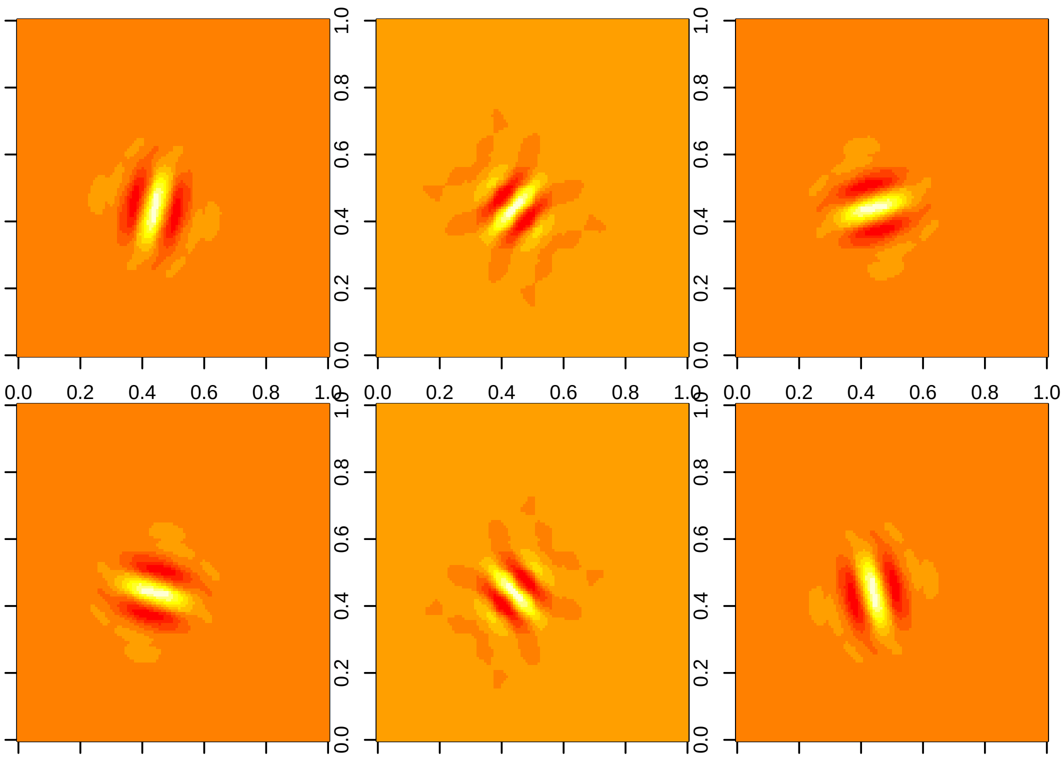
Complex wavelet transform

- In 1D, consider 2 separate orthogonal wavelet transforms, carefully chosen such that they can be interpreted as a real and imaginary components (Kingsbury, 2001)



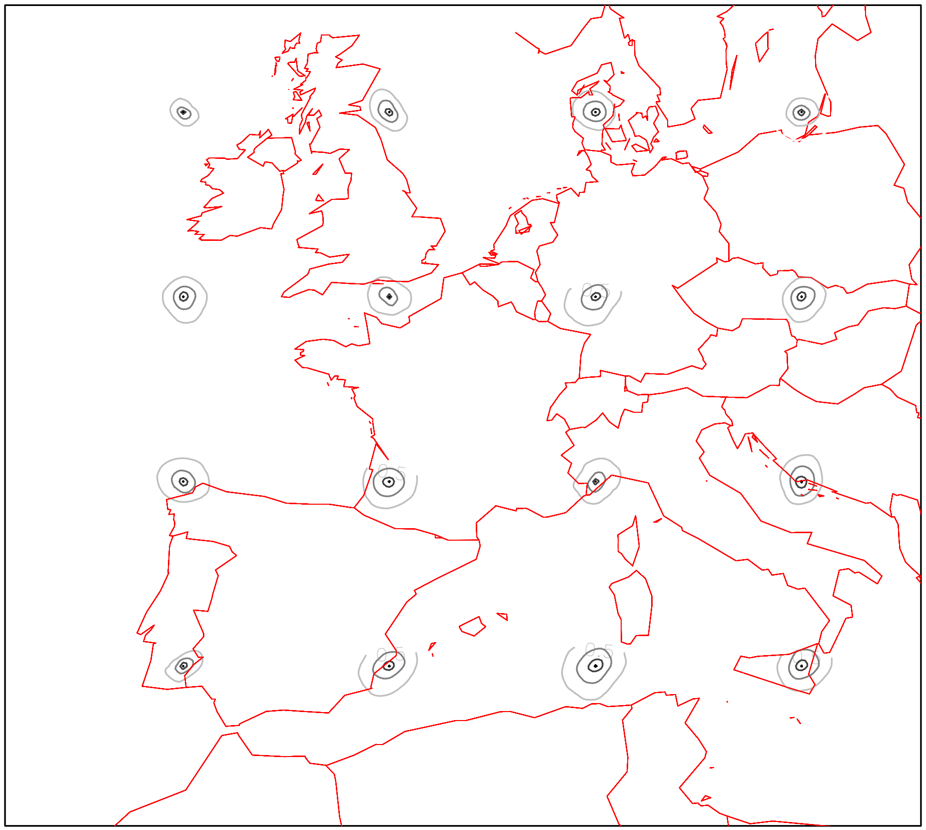
Complex wavelet transform

- In 2D, you need 4 different wavelet transforms and some linear combinations, to get a set of wavelets with clear orientations (only real part shown):



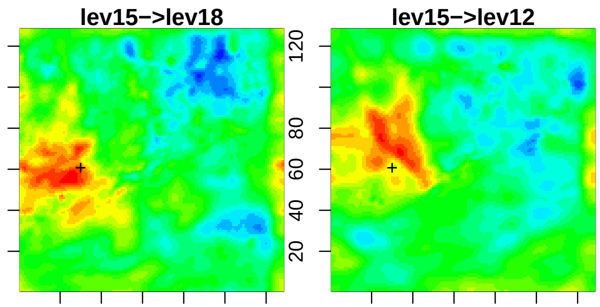
Complex wavelet transform

- If we use these directional wavelets to diagonalise B:



Vertical Structure

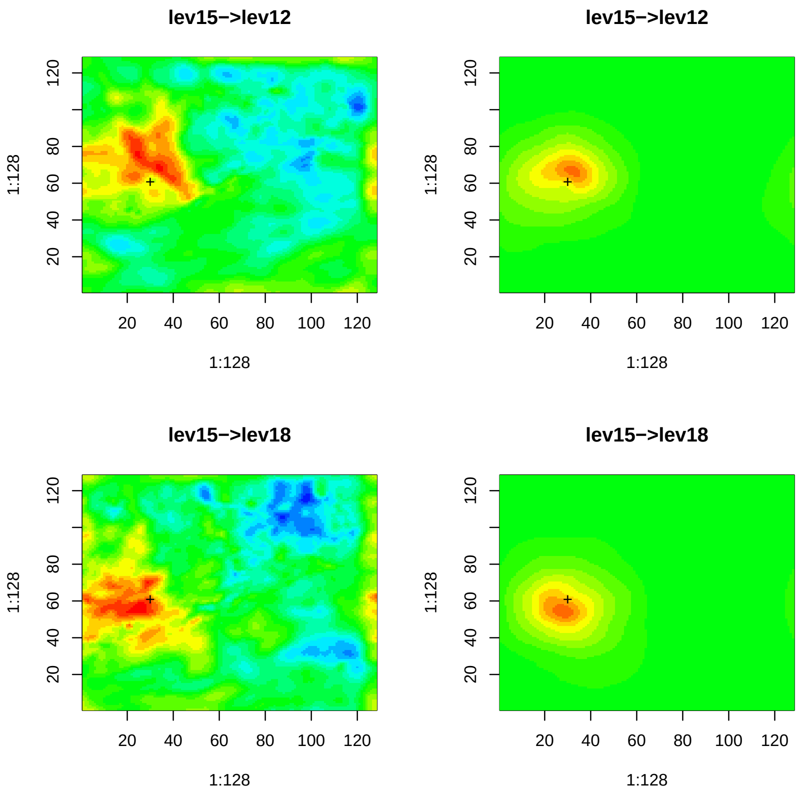
- How can we represent tilted structure functions winth a (block-) diagonal matrix?



- COMPLEX co-ordinates (e.g. Fourier)
- $var(A+iB, C+iD) = \overline{(A+iB)(C-iD)} = \overline{AC} + \overline{BD} + i(\overline{BC} - \overline{AD})$
- The phase of this (in general) complex covariance describes the tilt of the structure function.

Complex wavelet transform

- We can model tilted covariance functions between 2 levels with complex wavelets:



...largest scales?

Conclusions

- The anisotropy is much better modelled.
- Tilted vertical structure functions are possible.
- The wavelets are 4 $\times$ redundant (a so called *tight frame* in stead of a basis)
- Originally developped for motion detection.
- Some issues:
  - Border conditions: periodic or 0?
  - Most wavelets require domain size to be a power of 2, but there are some short-cuts.