# Comments on the EFB turbulence closure theory: turbulent fluxes of buoyant and passive scalars

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# 1. Introduction

Until present we considered the density stratification in the EFB turbulence closure (Zilitinkevich et al., 2012) in terms of the buoyancy flux $F_b$, which was taken proportional to the potential temperature flux $F_z^{(\theta)}$. In the wet air, $F_b$ depends also on the vertical turbulent flux of specific humidity $F_q$, so that $F_b \equiv \langle bw \rangle = F_z^{(\theta)}\beta + 0.61g\,F_q$, where $g$ is the acceleration of gravity, $\beta = g/T_0$ is the buoyancy parameter, $T_0$ is reference value of absolute temperature ($T_0^{-1}$ is the thermal expansion coefficient for ideal gas), $w$, $\theta$, $q$ and $b = \beta\theta + 0.61g\,q$ are fluctuations of vertical velocity, potential temperature, specific humidity (the mass of the water vapour per unit mass of fluid) and buoyancy, respectively, and angle brackets denote averaging. Furthermore, condensation of the water vapour or evaporation of droplets of liquid water suspended in the air affect the air temperature and vice versa. To determine separately $F_z^{(\theta)}$, $F_q$ and the turbulent flux of water droplets, we generalize the EFB closure accounting for essential interdependence of the temperature, humidity and liquid-water content.

We denote the actual values of the above listed meteorological parameters by the upright capital letters: $\mathrm{W}$, $\Theta$, Q and B, and the mean values by the same letters in Italic: $W$, $\mathit{\Theta}$, $Q$ and $B$ (so that $\mathrm{W} = W + w$, etc.). Similarly we denote the actual, mean and fluctuation values of the specific content of liquid water by A, $A$ and $a$. The mean potential temperature $\mathit{\Theta}$ is defined as $\mathit{\Theta} = T(P_0/P)^{1-1/\gamma}$, where $T$ is the absolute temperature, $P$ is the pressure, $P_0$ is its reference value, and $\gamma = c_p/c_v$ =1.41 is the specific heats ratio.

The fluxes $F_z^{(\theta)}, F_q$ and $F_a$ appear in the Reynolds-averaged equations for the mean-flow potential temperature $\mathit{\Theta}$, specific humidity $Q$ and liquid water content $A$:

$$\frac{D\mathit{\Theta}}{Dt} = -\frac{\partial F_z^{(\theta)}}{\partial z} + J + \frac{\lambda}{c_p} m, \qquad (2.1)$$

$$\frac{DQ}{Dt} = -\frac{\partial F_q}{\partial z} - m, \qquad (2.2a)$$

$$\frac{DA}{Dt} = -\frac{\partial F_a}{\partial z} + m, \qquad (2.2b)$$

where $J$ is the rate of heating/cooling due to the radiation heat transfer, $\lambda$ is the latent heat of condensation, $c_p$ is the specific heat, and $m$ is the rate of change of specific humidity due to evaporation/condensation. Clearly, $m \neq 0$ only in the presence of droplets of liquid water.

In the conditions of the thermodynamic equilibrium, $Q$ is maintained at the saturation value: $Q = Q_m \equiv (R/R_\mathrm{w})e(T)/P$, where $R$ and $R_\mathrm{w}$ are the gas constants of the dry air and the water vapour, and $e(T)$ is the partial pressure of the saturated water vapour at the temperature *T*, determined by the Clausius-Clapeyron equation: $de/e = (\lambda/R)dT/T^2$. Then *Q* depends only on $\Theta$ and *P*.

In the wet atmosphere the momentum equation becomes:

$$\frac{D\mathbf{V}}{Dt} = -\frac{\nabla \mathrm{P}}{\rho} + \nu\,\Delta\mathbf{V} + \beta\,\Theta_{\mathrm{v}}\,\mathbf{e}, \tag{2.3}$$

and the thermodynamic equations for the potential temperature, the specific humidity and the specific liquid water content become:

$$\frac{D\Theta}{Dt} = \kappa\,\Delta\Theta + \frac{\lambda}{c_p} m(\Theta,\mathrm{Q}), \tag{2.4}$$

$$\frac{D\mathrm{Q}}{Dt} = \kappa_Q\,\Delta\mathrm{Q} - m(\Theta,\mathrm{Q}), \tag{2.5a}$$

$$\frac{D\mathrm{A}}{Dt} = \kappa_A\,\Delta\mathrm{A} + m(\Theta,\mathrm{Q}), \tag{2.5b}$$

where $\mathbf{V} = \mathbf{U} + \mathbf{u}$ is the actual velocity [consisted of the mean $\mathbf{U} = (U_1, U_2, U_3) = (U, V, W)$ and the fluctuation $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ velocities], $D/Dt = \partial/\partial t + \mathrm{V}_i \partial/\partial x_i$, $\Theta_{\mathrm{v}} = \Theta + (\mu_{\mathrm{d}}/\mu_{\mathrm{w}} - 1)T_0\,\mathrm{Q} = \Theta + 0.61\,T_0\,\mathrm{Q}$ is the virtual temperature, $\mu_{\mathrm{d}}$ and $\mu_{\mathrm{w}}$ are the molar masses of the dry air and the water vapour, $\nu$ is kinematic viscosity, $\kappa$ is heat conductivity, $\kappa_Q$ and $\kappa_A$ are diffusivities for the water vapour and water droplets, respectively. We take into account that the condensation/evaporation rate *m* depends on temperature and humidity. Typical atmospheric flows are characterised by very low Mach numbers. Therefore, analysing turbulent statistics associated with the temperature, humidity and liquid-water content, the dependence of *m* on the atmospheric pressure can be neglected.

Equations (2.4) and (2.5) can be rewritten in terms of $\Theta_{\mathrm{v}}$ and the equivalent temperatures $\Theta_{\mathrm{e1}} = \Theta + \lambda\,\mathrm{Q}/c_p$ and $\Theta_{\mathrm{e2}} = \Theta - \lambda\,\mathrm{A}/c_p$:

$$\frac{D\Theta_{\mathrm{v}}}{Dt} = \kappa\,\Delta\Theta_{\mathrm{v}} + m_{\mathrm{v}}(\Theta_{\mathrm{v}}, \Theta_{\mathrm{e1}}), \tag{2.6}$$

$$\frac{D\Theta_{\mathrm{e1}}}{Dt} = \kappa\,\Delta\Theta_{\mathrm{e1}}, \tag{2.7a}$$

$$\frac{D\Theta_{\mathrm{e2}}}{Dt} = \kappa\,\Delta\Theta_{\mathrm{e2}}, \tag{2.7b}$$

where

$$m_{\mathrm{v}}(\Theta_{\mathrm{v}},\Theta_{\mathrm{ei}}) \equiv \left[\frac{\lambda}{c_p} - \left(\frac{\mu_{\mathrm{d}}}{\mu_{\mathrm{w}}} - 1\right) T_0\right] m(\Theta, q) . \tag{2.8}$$

For simplicity we took $\kappa_Q \approx \kappa_A \approx \kappa$. Subtracting the averaged version of Eqs. (2.6), (2.7) from the original equations yields the following equations for the fluctuations $\theta_{\mathrm{v}}$, $\theta_{\mathrm{e1}}$ and $\theta_{\mathrm{e2}}$:

$$\frac{D\theta_{\mathrm{v}}}{Dt} = \kappa \Delta \theta_{\mathrm{v}} - w\frac{N^2}{\beta} + \frac{\partial m_{\mathrm{v}}}{\partial \Theta_{\mathrm{v}}}\theta_{\mathrm{v}} + \frac{\partial m_{\mathrm{v}}}{\partial \Theta_{ei}}\theta_{\mathrm{ei}} - (\mathbf{u}\cdot\nabla)\theta_{\mathrm{v}} + \langle(\mathbf{u}\cdot\nabla)\theta_{\mathrm{v}}\rangle , \tag{2.9}$$

$$\frac{D\theta_{\mathrm{ei}}}{Dt} = \kappa \Delta \theta_{\mathrm{ei}} - w\frac{N_{ei}^2}{\beta} - (\mathbf{u}\cdot\nabla)\theta_{\mathrm{ei}} + \langle(\mathbf{u}\cdot\nabla)\theta_{\mathrm{ei}}\rangle , \tag{2.10}$$

where *i*=1,2, and

$$N^2 = \beta\,\frac{\partial \Theta_{\mathrm{v}}}{\partial z}, \qquad N_{ei}^2 = \beta\,\frac{\partial \Theta_{\mathrm{ei}}}{\partial z} \tag{2.11}$$

are the Brunt-Väisälä frequencies based on the virtual and the equivalent temperatures.

It follows from the definition of the virtual and the equivalent temperatures that

$$\mathrm{Q} = \left(\frac{\lambda}{c_p} - \frac{\mu_{\mathrm{d}} - \mu_{\mathrm{w}}}{\mu_{\mathrm{w}}}\right)^{-1} \left(\Theta_{\mathrm{e1}} - \Theta_{\mathrm{v}}\right), \tag{2.12}$$

$$\Theta = \Theta_{\mathrm{e1}} - \frac{\lambda}{c_p}\mathrm{Q}, \tag{2.13}$$

$$\mathrm{A} = \frac{c_p}{\lambda}\left(\Theta - \Theta_{\mathrm{e2}}\right) . \tag{2.14}$$

In the next section we employ Eqs. (2.9) and (2.10) to revise the basic equations of the EFB closure.

## 2. Budget equations for second moments accounting for condensation/evaporation

The budget equations for the turbulent kinetic energy (TKE) $E_K = \frac{1}{2}\langle \mathbf{u}^2\rangle$; the "energy" of the virtual-temperature fluctuations $E_\theta = \frac{1}{2}\langle \theta_{\mathrm{v}}^2\rangle$; the virtual-temperature flux $\mathbf{F}^{(\mathrm{v})} = \langle \mathbf{u}\,\theta_{\mathrm{v}}\rangle$;

and the second moments representing the "energy" of the equivalent-temperature fluctuations $E_{e1} = \frac{1}{2}\langle \theta_{e1}^2 \rangle$ and $E_{e2} = \frac{1}{2}\langle \theta_{e2}^2 \rangle$; the equivalent-temperature fluxes $\mathbf{F}^{(e1)} = \langle \mathbf{u}\, \theta_{e1} \rangle$ and $\mathbf{F}^{(e2)} = \langle \mathbf{u}\, \theta_{e2} \rangle$; and the cross-correlations of the virtual and equivalent temperature fluctuations $E_{c1} = \langle \theta_{\rm v}\, \theta_{e1} \rangle$ and $E_{c2} = \langle \theta_{\rm v}\, \theta_{e2} \rangle$ read:

$$\frac{DE_K}{Dt} + \frac{\partial}{\partial z}\Phi_K = -\tau_{i3}\frac{\partial U_i}{\partial z} + \beta F_z^{(\rm v)} - \varepsilon_K\,, \tag{3.1}$$

$$\frac{DE_\theta}{Dt} + \frac{\partial}{\partial z}\Phi_\theta = -F_z^{(\rm v)}\frac{N^2}{\beta} + 2\frac{\partial m_{\rm v}}{\partial \Theta_{\rm v}} E_\theta + \frac{\partial m_{\rm v}}{\partial \Theta_{ei}} E_{ci} - \varepsilon_\theta\,, \tag{3.2}$$

$$\frac{DF_z^{(\rm v)}}{Dt} + \frac{\partial}{\partial z}\Phi_z^{(F)} = -2E_z\frac{N^2}{\beta} + 2\beta E_\theta - \frac{1}{\rho_0}\left\langle \theta_{\rm v}\frac{\partial p}{\partial z}\right\rangle + \frac{\partial m_{\rm v}}{\partial \Theta_{\rm v}} F_z^{(\rm v)} + \frac{\partial m_{\rm v}}{\partial \Theta_{ei}} F_z^{(ei)} - \varepsilon_F\,, \tag{3.3}$$

$$\frac{DE_{ei}}{Dt} + \frac{\partial}{\partial z}\Phi_{ei} = -F_z^{(ei)}\frac{N_{ei}^2}{\beta} - \varepsilon_{ei}\,, \tag{3.4}$$

$$\frac{DF_z^{(ei)}}{Dt} + \frac{\partial}{\partial z}\Phi_z^{(ei)} = -2E_z\frac{N_{ei}^2}{\beta} + \beta E_{ci} - \frac{1}{\rho_0}\left\langle \theta_{ei}\frac{\partial p}{\partial z}\right\rangle - \varepsilon_F^{(ei)}\,, \tag{3.5}$$

$$\frac{DE_{ci}}{Dt} + \frac{\partial}{\partial z}\Phi_{ci} = -\left(F_z^{(ei)} N^2 + F_z^{(\rm v)} N_{ei}^2\right)\frac{1}{\beta} + \frac{\partial m_{\rm v}}{\partial \Theta_{\rm v}} E_{ci} + 2\frac{\partial m_{\rm v}}{\partial \Theta_{ei}} E_{ei} - \varepsilon_{ci}\,, \tag{3.6}$$

In Eqs. (3.4)-(3.6) $i$=1,2. Here, $\tau_{ij} = \langle u_i u_i \rangle$ is the Reynolds stress, $\Phi_K$, $\Phi_\theta$ and $\Phi_z^{(F)}$ are the third-order moments representing the turbulent transports of the TKE, of the "energy" of the virtual temperature fluctuations and of the virtual-temperature flux:

$$\Phi_K = \frac{1}{\rho_0}\langle p\, w \rangle + \frac{1}{2}\langle u^2 w \rangle, \tag{3.7a}$$

$$\Phi_\theta = \frac{1}{2}\langle \theta_{\rm v}^2\, w \rangle, \tag{3.7b}$$

$$\Phi_z^{(F)} = \frac{1}{2\rho_0}\langle p\theta_{\rm v} \rangle + \langle w^2 \theta_{\rm v} \rangle, \tag{3.7c}$$

and $\rho_0$ is the reference value of the air density.

The third-order moments:

$$\Phi_{ei} = \frac{1}{2}\left\langle \theta_{ei}^2 \; w \right\rangle, \tag{3.8a}$$

$$\Phi_z^{(ei)} = \frac{1}{2\rho_0}\left\langle p\theta_{ei}\right\rangle + \left\langle w^2\theta_{ei}\right\rangle, \tag{3.8b}$$

$$\Phi_{ci} = \left\langle \theta_{\mathrm{v}}\,\theta_{ei}\; w\right\rangle \tag{3.8c}$$

express turbulent transports of the "energy" of the equivalent-temperature fluctuations, the equivalent-temperature fluxes, and the cross-correlation between fluctuations of the virtual and equivalent temperatures.

The terms $\varepsilon_K$, $\varepsilon_\theta$ and $\varepsilon_F$ are essentially positive operators representing the dissipation rates for $E_K$, $E_\theta$, $F_z^{(\mathrm{v})}$, $E_{ei}$, $F_z^{(ei)}$ and $E_{ci}$, respectively. Following Kolmogorov (1941), they are taken proportional to the ratios of the dissipating moments to the dissipation time scale, $t_T$ :

$$\varepsilon_K = \nu\left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}\right\rangle = \frac{E_K}{t_T}, \tag{3.9a}$$

$$\varepsilon_\theta = -\kappa\left\langle \theta_{\mathrm{v}}\;\Delta\,\theta_{\mathrm{v}}\right\rangle = \frac{E_\theta}{C_P\; t_T}, \tag{3.9b}$$

$$\varepsilon_F = -\kappa\left(\left\langle w\,\Delta\,\theta_{\mathrm{v}}\right\rangle + \mathrm{Pr}\left\langle \theta_{\mathrm{v}}\;\Delta\, w\right\rangle\right) = \frac{F_z^{(\mathrm{v})}}{C_F\; t_T}, \tag{3.9c}$$

$$\varepsilon_{ei} = -\kappa\left\langle \theta_{ei}\;\Delta\,\theta_{ei}\right\rangle = \frac{E_{ei}}{C_P^{(ei)}\; t_T}, \tag{3.10a}$$

$$\varepsilon_F^{(ei)} = -\kappa\left(\left\langle w\,\Delta\,\theta_{ei}\right\rangle + \mathrm{Pr}\left\langle \theta_{ei}\;\Delta\, w\right\rangle\right) = \frac{F_z^{(ei)}}{C_F^{(ei)}\; t_T}, \tag{3.10b}$$

$$\varepsilon_{ci} = -\kappa\left(\left\langle \theta_{\mathrm{v}}\;\Delta\,\theta_{ei}\right\rangle + \left\langle \theta_{ei}\;\Delta\,\theta_{\mathrm{v}}\right\rangle\right) = \frac{E_{ci}}{C_{ci}\; t_T}, \tag{3.10c}$$

where $\kappa$ is the temperature conductivity, $\mathrm{Pr} = \nu/\kappa$ is the Prandtl number, $C_K$, $C_P$, $C_F$, $C_P^{(ei)}$, $C_F^{(ei)}$ and $C_{ci}$ are dimensionless constants.

As demonstrated by Zilitinkevich et al. (2007), the term $-\rho_0^{-1}\left\langle \theta\,\partial p/\partial z\right\rangle$ in the budget equation for $F_z$ is essentially negative and scales as $\beta\left\langle \theta^2\right\rangle$, so that the pair of term

$\beta\langle\theta^2\rangle-\rho_0^{-1}\langle\theta\,\partial p/\partial z\rangle$ is expressed as $C_\theta\,\beta\langle\theta^2\rangle$, where $C_\theta<1$ is empirical dimensionless constant. We apply the same parameterization to the analogous terms in Eqs. (3.3) and (3.5):

$$\left\langle\theta_{\mathrm{v}}\left(\beta\theta_{\mathrm{v}}-\rho_0^{-1}\,\partial p/\partial z\right)\right\rangle=C_\theta\,\beta\left\langle\theta_{\mathrm{v}}^2\right\rangle, \tag{3.11a}$$

$$\left\langle\theta_{\mathrm{ei}}\left(\beta\theta_{\mathrm{v}}-\rho_0^{-1}\,\partial p/\partial z\right)\right\rangle=C_\theta^{(ei)}\beta\,E_{ci}\,, \tag{3.11b}$$

which yields the simplified versions of these equations:

$$\frac{DF_z}{Dt}+\frac{\partial}{\partial z}\Phi_z^{(F)}=-2(E_z-C_\theta E_P)\frac{N^2}{\beta}+\frac{\partial m_{\mathrm{v}}}{\partial\Theta_{\mathrm{v}}}F_z^{(\mathrm{v})}+\frac{\partial m_{\mathrm{v}}}{\partial\Theta_{ei}}F_z^{(ei)}-\varepsilon_F\,, \tag{3.12}$$

$$\frac{DF_z^{(ei)}}{Dt}+\frac{\partial}{\partial z}\Phi_z^{(ei)}=-2E_z\frac{N_{ei}^2}{\beta}+C_\theta^{(ei)}\beta\,E_{ci}-\varepsilon_F^{(ei)}\,, \tag{3.13}$$

where $E_P=(\beta/N)^2E_\theta$ is the turbulent potential energy, and $C_\theta^{(ei)}<1$ is empirical dimensionless constant.

## 3. Steady-state regime of turbulence

In the steady-state, the left hand sides (l.h.s.) of the above budget equations turn into zero, and the equations become algebraic. Then Eqs. (3.4), (3.6) and (3.13) yield:

$$F_z^{(ei)}=-2\Psi_F^{(ei)}A_z\,E_K\,t_T\,\frac{N_{ei}^2}{\beta}\left(1+C_\theta^{(ei)}\,\Psi_{ci}\,\frac{\beta\,F_z^{(\mathrm{v})}\,t_T}{2\,A_z\,E_K}\right), \tag{4.1}$$

$$E_{ei}=-C_p^{(ei)}\,t_T\,F_z^{(ei)}\,\frac{N_{ei}^2}{\beta}\,, \tag{4.2}$$

$$E_{ci}=-\Psi_{ci}\,t_T\,\frac{N_e^2}{\beta}\left[F_z^{(\mathrm{v})}+F_z^{(ei)}\left(\frac{N^2}{N_{ei}^2}+2C_p^{(ei)}t_T\,\frac{\partial m_{\mathrm{v}}}{\partial\Theta_{ei}}\right)\right], \tag{4.3}$$

where $\Psi_{ci}$, $\Psi_F^{(ei)}$ are combinations introduced to make shorter further relations:

$$\Psi_{ci}=\left(C_{ci}^{-1}-t_T\,\frac{\partial m_{\mathrm{v}}}{\partial\Theta_{\mathrm{v}}}\right)^{-1}, \tag{4.4a}$$

$$\Psi_F^{(ei)} = \left[ C_{ci}^{-1} + C_\theta^{(ei)} \Psi_{ci} t_T^2 N^2 \left( 1 + 2 t_T C_p^{(ei)} \frac{N_{ei}^2}{N^2} \frac{\partial m_{\mathrm{v}}}{\partial \Theta_{ei}} \right) \right]^{-1}, \tag{4.4b}$$

and $A_z = E_z / E_K$ is the share of the "vertical energy".

In the turbulent kinetic energy (TKE) budget Equation (3.1), the first term on the r.h.s. is the rate of the TKE production: $-\tau_{i3} \partial U_i / \partial z = \tau S$, where $\tau$ and $S$ are absolute values of the vectors $\boldsymbol{\tau}$ and $\partial \mathbf{U} / \partial z$, and the second term $\beta F_z^{(\mathrm{v})}$ is the rate of conversion of the TKE into the turbulent potential energy (TPE) $E_P = (\beta / N)^2 E_\theta$. The ratio of these terms is called flux Richardson number

$$Ri_f \equiv \frac{-\beta F_z^{(\mathrm{v})}}{\tau S}. \tag{4.5}$$

Using these notations, Eq. (4.1) becomes

$$F_z^{(ei)} = -K_M \frac{\Psi_F^{(ei)}}{C_\tau \beta} \left( 1 - C_\theta^{(ei)} \Psi_{ci} \frac{Ri_f}{2 A_z (1 - Ri_f)} \right) N_{ei}^2, \tag{4.6}$$

where the eddy viscosity $K_M$, as well as $A_z$, are precisely the same as in the dry atmosphere:

$$K_M = 2 C_\tau A_z E_K t_T, \tag{4.7}$$

$$A_z = \frac{C_r (1 - 2 C_0 Ri_f / R_\infty)(1 - Ri_f) - 3 Ri_f}{(1 - Ri_f)\{3 + C_r [3 - 2(1 + C_0) Ri_f / R_\infty]\}}. \tag{4.8}$$

The dimensionless empirical constants $C_\tau$, $C_r$, $C_0$, $C_\tau$, $R_\infty$ have been already estimated by Zilitinkevich et al., (2012).

The steady-state versions of Eqs. (3.2) and (3.12) yield:

$$E_\theta = -\Psi_P t_T \left( F_z^{(\mathrm{v})} \frac{N^2}{\beta} - \frac{\partial m_{\mathrm{v}}}{\partial \Theta_{ei}} E_{ci} \right), \tag{4.9}$$

$$F_z^{(\mathrm{v})} = -\Psi_F t_T \left( 2(E_z - C_\theta E_P) \frac{N^2}{\beta} - \frac{\partial m_{\mathrm{v}}}{\partial \Theta_{ei}} F_z^{(ei)} \right), \tag{4.10}$$

where $\Psi_P$ and $\Psi_F$ are combination:

$$\Psi_P = \left( C_P^{-1} - 2t_T \frac{\partial m_\mathrm{v}}{\partial \Theta_\mathrm{v}} \right)^{-1}, \tag{4.11}$$

$$\Psi_F = \left( C_F^{-1} - t_T \frac{\partial m_\mathrm{v}}{\partial \Theta_\mathrm{v}} \right)^{-1}. \tag{4.12}$$

Equations (4.9)-(4.10) in combinations with Eq. (4.3) yield:

$$\frac{E_P}{E_K} = \frac{\Psi_P \, Ri_f}{1 - Ri_f} \left\{ 1 + \Psi_{ci} \, t_T \frac{\partial m_\mathrm{v}}{\partial \Theta_{ei}} \frac{N_{ei}^2}{N^2} \left[ 1 + 2A_z \, t_T^2 \, N^2 \, \Psi_F \Psi \left( \frac{1 - Ri_f}{Ri_f} - \frac{C_\theta^{(ei)} \Psi_{ci}}{2A_z} \right) \right] \right\}, \tag{4.13}$$

$$F_z^{(\mathrm{v})} = -K_M \frac{N^2 \Psi_F}{\beta C_\tau} \left[ 1 - \frac{C_\theta \, E_P}{A_z \, E_K} + \frac{\partial m_\mathrm{v}}{\partial \Theta_{ei}} \frac{\Psi_F^{(ei)} t_T}{N^2} \left( 1 - \frac{C_\theta^{(ei)} \Psi_c \, Ri_f}{2A_z (1 - Ri_f)} \right) N_{ei}^2 \right] \tag{4.14}$$

$$\equiv -K_H \frac{N^2}{\beta},$$

where $K_H$ is (by definition) the eddy conductivity, and $\Psi$ is one more combination:

$$\Psi = 1 + 2C_P^{(ei)} \, t_T \frac{\partial m_\mathrm{v}}{\partial \Theta_{ei}} \frac{N_{ei}^2}{N^2}. \tag{4.15}$$

Equations (4.7) and (4.10) yield the relation:

$$t_T^2 \, N^2 = \frac{Ri}{2C_\tau \, A_z (1 - Ri_f)}, \tag{4.16}$$

where $Ri = N^2 / S^2$ is the gradient Richardson number. Substituting Eq. (4.16) it (4.14) yields the expression linking the turbulent Prandtl number $Pr_T$, the flux Richardson number $Ri_f$, and the gradient Richardson number $Ri$:

$$Pr_T \equiv \frac{K_M}{K_H} \equiv \frac{Ri}{Ri_f} = \frac{C_\tau}{\Psi_F} \left[ 1 - \frac{C_\theta \, E_P}{A_z \, E_K} + \Psi_F^{(ei)} \, t_T \frac{\partial m_\mathrm{v}}{\partial \Theta_{ei}} \frac{N_e^2}{N^2} \left( 1 - \frac{C_\theta^{(ei)} \Psi_{ci} \, Ri_f}{2A_z (1 - Ri_f)} \right) \right]^{-1}. \tag{4.17}$$

It follows that the dependencies of $Ri_f$ or $Pr_T$ on $Ri$, playing the key role in the EFB closure, are generally affected by the processes of condensation/evaporation. With this remark, the dependencies of the dimensionless turbulent fluxes of momentum and heat on $Ri_f$ have precisely the same form as in dry air.

Using Eqs. (2.12)-(2.14) we find link between the fluxes $F_z^{(\theta)}, F_q$ , $F_a$ and the fluxes of the virtual and the equivalent temperatures:

$$F_q = \left( \frac{\mu_{\rm d} - \mu_{\rm w}}{\mu_{\rm w}} T_0 - \frac{\lambda}{c_p} \right)^{-1} \left( F_z^{(\rm v)} - F_z^{(e1)} \right), \tag{4.18}$$

$$F_z^{(\theta)} = F_z^{(e1)} - \frac{\lambda}{c_p} F_q , \tag{4.19}$$

$$F_a = \frac{c_p}{\lambda} \left( F_z^{(\theta)} - F_z^{(e2)} \right). \tag{4.20}$$

Taking partial derivatives of *m* equal to zero yields the fluxes of the equivalent $F_z^{(ei)}$ and virtual $F_z^{(\rm v)}$ temperature fluctuations:

$$F_z^{(ei)} = -K_M \, \frac{C_{ci} N_{ei}^2}{C_\tau \, \beta} \left( 1 - \frac{C_\theta^{(ei)} \, C_{ci} \, Ri_f}{2 \, A_z \, (1 - Ri_f)} \right) \left( 1 + \frac{C_\theta^{(ei)} \, C_{ci}^2 \, Ri}{2 \, A_z \, C_\tau (1 - Ri_f)} \right)^{-1} , \tag{4.21}$$

$$F_z^{(\rm v)} = -K_M \, \frac{N^2 \, C_F}{\beta C_\tau} \left[ 1 - \frac{C_\theta \, C_P \, Ri_f}{A_z \, (1 - Ri_f)} \right], \tag{4.22}$$

where *i*=1,2. Substituting Eqs. (4.21)-(4.22) into Eqs. (4.18)-(4.20) allows us to determine the fluxes $F_z^{(\theta)}, F_q$ and $F_a$ .

# Conclusions

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