Vertical Finite Elements in NH dynamical core

Implementation with mass coordinate based system

Regional Cooperation for
Limited Area Modeling in Central Europe

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nwp central europe

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23rd ALADIN Workshop and HIRLAM All Staff Meeting 2013

Outline

1. Formulation
   - Prognostic variables
2. Vertical discretisation
   - Vertical levels and material boundaries
   - Finite elements with B-splines
   - Definition of full level *A* and *B*
   - Linear system - discretisation
   - Nonlinear system - discretization
3. Experiments
   - 2D idealized experiments
   - 3D adiabatic experiment

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Dynamics formulation

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- prognostic variables
  - grid point space

$$u, v, T, ln(\pi_s), \hat{q}, gw$$

  - spectral space

$$D, \zeta, T, ln(\pi_s), \hat{q}, d_4$$

with

$$d_4 = \frac{p}{mRT}\frac{\partial gw}{\partial \eta} + \frac{p}{mRT}\vec{\nabla}\phi\frac{\partial \vec{v}}{\partial \eta}$$

$$\hat{q} = ln\left(\frac{p}{\pi}\right)$$

- vertical coordinate

$$\pi(\eta) = A(\eta) + B(\eta)\pi_s$$

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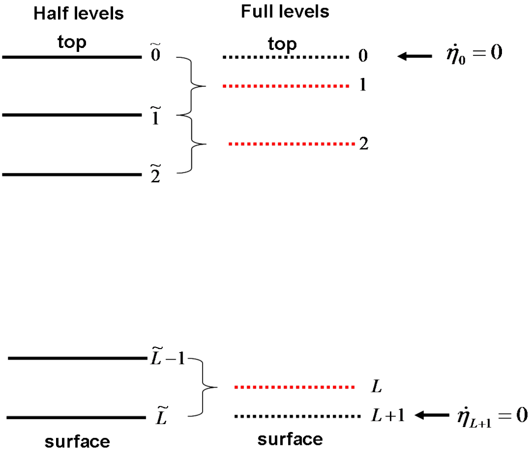
1. full levels - all prognostic quantities except *gw*
2. half levels - *gw*, π



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B-spline basis functions

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Problem:
Having data points $[\pi_i, f(\pi_i)]$ on full levels and material boundaries in mass coordinate system, we would like to interpolate/approximate this data points with parametric B-spline curve as

$$S[\eta, f(\eta)] = \sum_{i=0}^{L+1} \left[\hat{\eta}_i, \hat{f}_i\right] B_i(\eta) \quad (1)$$

To do this approximation we have to

1. define knots to constructs spline basis B_i with order C (knots - vector of η values used to construct B-spline basis),
2. determine values of parameter $\eta_i = \eta(\pi_i^*)$ in data points,
3. determine spline curve control points $\left[\hat{\eta}_i, \hat{f}_i\right]$.

Definition knots to construct B-spline basis

For knots definition we adopted the centripetal method

1. Half level parameter values

$$\eta_{\tilde{i}} = \frac{\sum_{l=1}^{i} (\pi^*_{\tilde{l}} - \pi^*_{\widetilde{l-1}})^{\alpha}}{\sum_{i=1}^{L} (\pi^*_{\tilde{l}} - \pi^*_{\widetilde{l-1}})^{\alpha}},$$

2. Full level parameter values

$$\eta_0 = 0, \eta_i = \frac{\eta_{\tilde{i}} + \eta_{\widetilde{i-1}}}{2}, \eta_{L+1} = 1$$

3. vector of knots of lenth $K = L + 2 + c$

$$k_i = \begin{cases} 0 & : i = 1, c \\ \eta_{i+1-c} & : i = c+1, L+2 \\ 1 & : i = L+3, L+2+c \end{cases}$$

Definition of knots to construct B-spline basis

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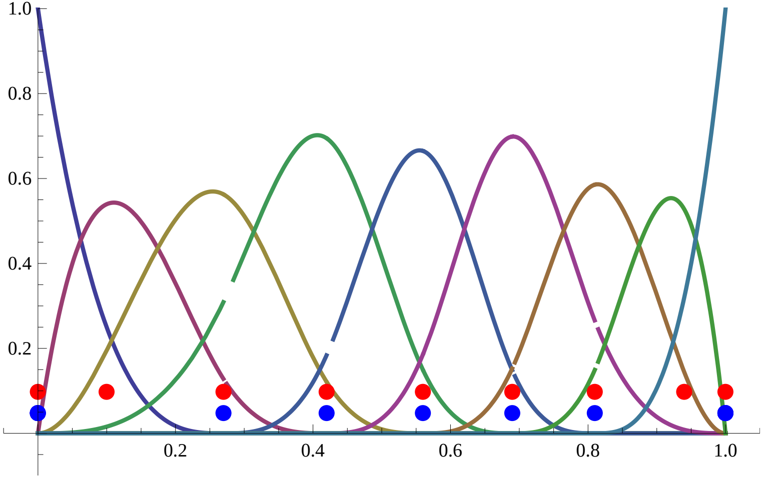
The B-spline basis functions are constructed using DeBoor's algorithm. Here is an example for 7 vertical full levels.



red - full level parameter η, blue - knots

The choice of full/half level parameter

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Having B-spline basis we are free to choose the values of parameter η on model full levels. This affects the shape of B-spline curve.We tested following methods:

1. the model method

$$\eta_{\tilde{l}} = \frac{A_{\tilde{l}}}{p_{ref}} + B_{\tilde{l}} \quad \eta_l = \frac{\eta_{\tilde{l}} + \eta_{\tilde{l}-1}}{2}$$

2. the centripetal method (the same as for knots construction)
3. the universal method (values computed from knots)

$$\eta_i = \frac{(k_{i+2} + ... + k_{i+c})}{(c-1)} \quad \eta_{\tilde{i}} = \frac{(\eta_i + \eta_{i+1})}{2}$$

B-spline curve control points

The control points $\left[\hat{\eta}_i, \hat{f}_i\right]$ in B-spline curve expression

$$S\left[\eta, f(\eta)\right] = \sum_{i=0}^{L+1} \left[\hat{\eta}_i, \hat{f}_i\right] B_i(\eta) \tag{2}$$

are determined from known L full level data points $[\eta_i, f_i]$ and 2 BCs on material boundaries (a priori known) assuming:

- interpolating curve (it passes through data points)

$$\left[\eta_k, f(\eta_k)\right] = \sum_{i=0}^{L+1} \mathbf{B_i}(\eta_\mathbf{k}) \left[\hat{\eta}_i, \hat{f}_i\right]$$

- variation diminishing approximation (full level are determined by universal method)

$$\left[\eta_k, f(\eta_k)\right] = \sum_{i=0}^{L+1} \mathbf{I_{ik}} \left[\hat{\eta}_i, \hat{f}_i\right]$$

B-spline curve examples - loosely resolved signal

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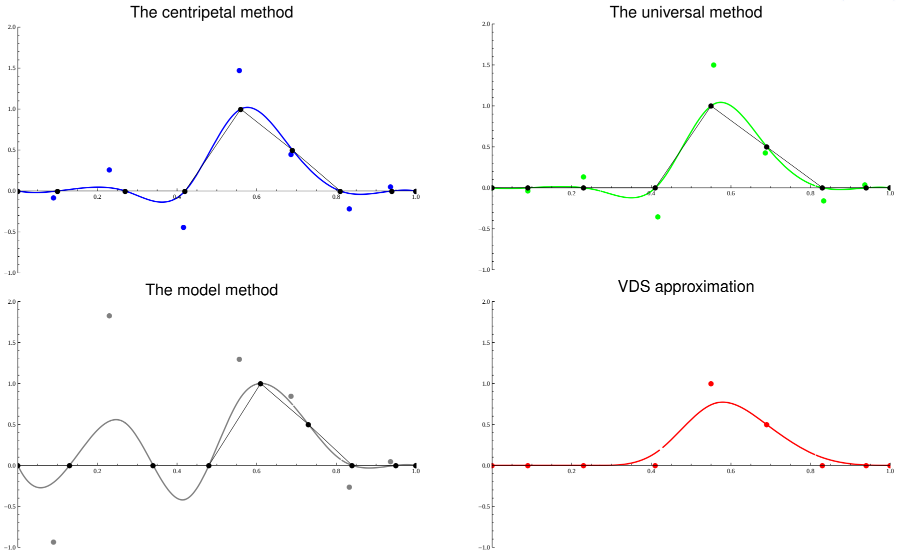


color solid lined - B-spline curve, black line - connected data points, color dots - control points

Vertical operators

Knowing control points $\hat{f}_i$ we could apply FE procedure (here shown first derivative operator):

$$\frac{\partial f(\eta)}{\partial \eta} = d(\eta) \quad \Rightarrow \quad \sum_{i=0}^{L+1} \hat{f}_i \frac{\partial B_i(\eta)}{\partial \eta} = \sum_{i=0}^{L+1} \hat{d}_i D_i(\eta)$$

using mean weighted residual approach with weighting functions $w_j = B_j$ we get

$$\sum_{i=0}^{L+1} \left[\int_{\mathbf{0}}^{\mathbf{1}} \frac{\partial \mathbf{B_i}(\eta)}{\partial \eta} \mathbf{w_j}(\eta) \mathbf{d}\eta \right] \hat{f}_i = \sum_{i=0}^{L+1} \left[\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{D_i}(\eta) \mathbf{w_j}(\eta) \mathbf{d}\eta \right] \hat{d}_i$$

Value of operator (derivative) is evaluated at location η_k as

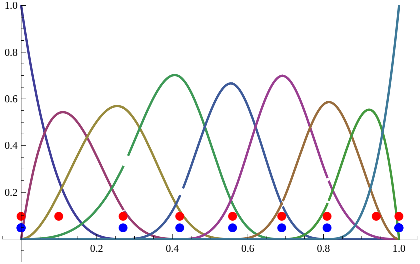
$$d(\eta_k) = \sum_{i=0}^{L+1} \mathbf{D_i}(\eta_\mathbf{k}) \hat{d}_i$$

Operators - boundary conditions

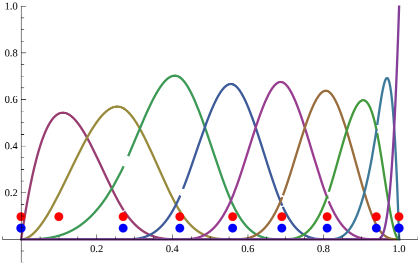
Newton or Dirichlet BCs are imposed on material boundaries:

- on input quantity directly (values of f_0 or $\frac{\partial f}{\partial \eta}_0$ at model top, resp. f_{L+1} or $\frac{\partial f}{\partial \eta}_{L+1}$ at model surface),
- on output - the basis functions are adjusted to satisfy required BCs.

B_i - no BC conditions



D_i - basis functions with 0 top BC



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Definition of full level A and B

We adopted the method from Hydrostatic VFE. From

$$\delta A_l = A_{\tilde{l}} - A_{\tilde{l}-1} \quad \delta B_l = B_{\tilde{l}} - B_{\tilde{l}-1} \quad \delta \eta_l = \eta_{\tilde{l}} - \eta_{\tilde{l}-1}$$

we could get full level values of A and B as

$$\left(\mathbf{K}\frac{\delta A}{\delta \eta}\right)_l = A_l \quad \left(\mathbf{K}\frac{\delta B}{\delta \eta}\right)_l = B_l$$

with conditions

$$\left(\mathbf{K}\frac{\delta A}{\delta \eta}\right)_{L+1} = 0 \quad \left(\mathbf{K}\frac{\delta B}{\delta \eta}\right)_{L+1} = 1$$

we iteratively rescale δB and δA to fullfill conditions. Then

- $\pi_l = A_l + B_l \pi_s$
- $m_l = \frac{\delta A_l}{\eta_l} + \frac{\delta B_l}{\eta_l} \pi_s$

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Linear system

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$$
\begin{aligned}
\frac{\partial D}{\partial t} &= -R\mathbf{G}^{*}\triangle T + RT^{*}(\mathbf{G}^{*} - 1)\triangle\hat{q} - RT^{*}\triangle q_s - \triangle\phi_s \\
\frac{\partial T}{\partial t} &= -\frac{RT^{*}}{C_v}(D + d_4) \\
\frac{\partial q_s}{\partial t} &= -\mathbf{N}^{*}D \\
\frac{\partial d_4}{\partial t} &= -\frac{g^2}{RT_a{}^{*}}\mathbf{L}^{*}\hat{q} \\
\frac{\partial \hat{q}}{\partial t} &= -\frac{C_p}{C_v}(D + d_4) + \mathbf{S}^{*}D.
\end{aligned}
\tag{3}
$$

Integral VFE operators **K** from model top and **P** from model surface are introduced

$$\int_0^\eta \psi d\eta = (\mathbf{K}\psi)_\eta \quad \int_\eta^1 \psi d\eta = (\mathbf{K}\psi)_1 - (\mathbf{K}\psi)_\eta = (\mathbf{P}\psi)_\eta$$

with BCs

operator	input BCs	output BCs
$\mathbf{K}\psi$	$\frac{\partial\psi}{\partial\eta}_0 = 0,\ \frac{\partial\psi}{\partial\eta}_{L+1} = 0$	$(\mathbf{K}\psi)_0 = 0$

linear integral operators

$$\begin{aligned} (\mathbf{S}^* X)_l &= \frac{1}{\pi_l^*} (\mathbf{K} m^* X)_l \\ (\mathbf{G}^* X)_l &= \left(\mathbf{P} \frac{m^*}{\pi^*} X\right)_l \\ (\mathbf{N}^* X)_l &= (\mathbf{S}^* X)_{L+1} \end{aligned}$$

Discrete laplacian term

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Continuos form is reformulated as

$$
\begin{aligned}
\mathbf{L}^* X &= \frac{1}{m^*}\frac{\partial}{\partial \eta}\left(\frac{{\pi^*}^2}{m^*}\right)\frac{\partial \hat{q}}{\partial \eta} + \left(\frac{\pi^*}{m^*}\right)^2 \frac{\partial^2 \hat{q}}{\partial \eta^2} \\
&= \frac{1}{m^*}\mathbf{D_1}\left(\frac{{\pi^*}^2}{m^*}\right)\mathbf{D_2}\hat{q} + \left(\frac{\pi^*}{m^*}\right)^2 \mathbf{DD}\hat{q}
\end{aligned}
$$

with boundary conditions

operator	input BCs	output BCs
$\mathbf{D_2}\hat{q}$	$\hat{q}_0 = 0, \frac{\partial \hat{q}}{\partial \eta}_{L+1} = 0$	$(\mathbf{D_2}\hat{q})_{L+1} = 0$
$\mathbf{DD}\hat{q}$	$\hat{q}_0 = 0, \frac{\partial \hat{q}}{\partial \eta}_{L+1} = 0$	$(\mathbf{DD}\hat{q})_{L+1} = 0$
$\mathbf{D_1}\left(\frac{{\pi^*}^2}{m^*}\right)$	$\left(\frac{{\pi^*}^2}{m^*}\right)_0 = 0, \left(\frac{{\pi^*}^2}{m^*}\right)_{L+1} = \left(\frac{{\pi_s^*}^2}{{m^*}_L}\right)$	-

Semi-implicit solver

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Algebraic elimination of all variables but d_4 is not feasible with VFE operators as

$$-\mathbf{G}^*\mathbf{S}^* + \mathbf{S}^* + \mathbf{G}^* - \mathbf{N}^* \neq 0 \quad => \quad \mathbb{C} \neq 0. \tag{4}$$

We finish the system of two equation (2Lx2L) for each horizontal wavenumber.

$$\begin{pmatrix} \mathbb{H} & \mathbb{F}\mathbb{C} \\ -\mathbb{B} & \mathbb{A} + \mathbb{C} \end{pmatrix} \begin{pmatrix} d \\ D \end{pmatrix} = \begin{pmatrix} \mathbb{A} & \mathbb{F} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} d^\bullet \\ D^\bullet \end{pmatrix}.$$

We have implemented

- the direct 2L solver (memory consuming, conflict with some other model keys)
- iterative solution around 1L solver (more in next Petra's talk)

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Nonlinear system of equations

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$$
\begin{aligned}
\frac{dgw}{dt} &= \frac{g^2}{m}\frac{\partial(p-\pi)}{\partial\eta} \\
\frac{du}{dt} &= -\frac{RT}{p}\frac{\partial p}{\partial x} - \left(\frac{1}{m}\frac{\partial p}{\partial\eta}\right)\frac{\partial\phi}{\partial x} + fv \\
\frac{dv}{dt} &= -\frac{RT}{p}\frac{\partial p}{\partial y} - \left(\frac{1}{m}\frac{\partial p}{\partial\eta}\right)\frac{\partial\phi}{\partial y} - fu \\
\frac{dT}{dt} &= -\frac{R}{C_v T} D_3 \\
\frac{\partial q_s}{\partial t} &= -\frac{1}{\pi_s}\int_0^1 \vec{\nabla}\left(m\vec{v}\right) d\eta \\
\frac{d\hat{q}}{dt} &= \frac{C_p}{C_v} D_3 - \frac{\omega}{\pi}
\end{aligned}
\tag{5}
$$

Vertical momentum equation

Having atmosphere in steady state we have

$$\frac{\partial X}{\partial t} = 0$$

every time step we perform transformation

at time t	at time $t \tilde{+} dt$ (explicit guess)
$gw^t = gw_s^t + \mathbf{T_i}\left[\frac{mRT}{p}(d_4 - \mathcal{X})\right]^t$	${d_4}^{t\tilde{+}dt} = \left[\frac{p}{mRT}\mathbf{T_d}\left(gw\right) + \mathcal{X}\right]^{t\tilde{+}dt}$

These must not influence steady state => ${d_4}^{t\tilde{+}dt} = {d_4}^t$. This requires

$$\mathbf{T_i}\mathbf{T_d} = \mathbf{Id}$$

This we are not able to ensure with VFE method and *gw* must be

- half level quantity
- $\mathbf{T_i}$ and $\mathbf{T_d}$ - FD operators.

Vertical momentum equation

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$$\frac{dgw}{dt} = \frac{g^2}{m}\frac{\partial(p-\pi)}{\partial\eta}$$

Since gw is a half level quantity the RHS must be evaluated at half levels

$$\left(\frac{dgw}{dt}\right)_{\tilde{l}} = \frac{g^2}{m_{\tilde{l}}}\mathbf{D_h}(p-\pi)_{\tilde{l}}$$

with

operator	input BCs (full levels)	output (half levels)
$\mathbf{D_h}(p-\pi)$	$(p-\pi)_0 = 0,\ (p-\pi)_{L+1} = (p-\pi)_L$	-

and

$$m_{\tilde{l}} = \frac{\pi_{l+1}-\pi_l}{\eta_{l+1}-\eta_l}$$

The pressure gradient force term

$$PGF = \left(\frac{1}{m}\frac{\partial p}{\partial \eta}\right)\vec{\nabla}\phi$$

is discretized on full model level l as

$$\left(\frac{1}{m}\frac{\partial p}{\partial \eta}\right)_l = \frac{p_l}{\pi_l} + \frac{p_l}{m_l}\left(\mathbf{D_1}\hat{q}\right)_l$$

with

operator	input BCs (full levels)	output (half levels)
$\mathbf{D_1}\hat{q}$	$\hat{q}_0 = 0,\ \hat{q}_{L+1} = \hat{q}_L$	-

and

$$\vec{\nabla}\phi_l = \vec{\nabla}\phi_s + \mathbf{P}\left[\frac{mRT}{p}\left(\frac{\vec{\nabla}m}{m} + \frac{\vec{\nabla}RT}{RT} - \frac{\vec{\nabla}p}{p}\right)\right]_l$$

Discretisation of D_3

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$$D_3 = D + d_4 = D + \frac{p}{mRT}\frac{\partial gw}{\partial \eta} - \frac{p}{mRT}\vec{\nabla}\phi\frac{\partial \vec{v}}{\partial \eta}$$

$\vec{\nabla}\phi$ is computed the same way as in the case of pressure gradient force term. The vertical derivative of wind is computed as

$$\left(\frac{\partial \vec{v}}{\partial \eta}\right)_l = \left(\mathbf{D_1}\vec{v}\right)_l$$

with

operator	input BCs (full levels)	output (half levels)
$\mathbf{D_1}\vec{v}$	$\vec{v}_0 = \vec{v}_1,\ \vec{v}_{L+1} = \vec{v}_L$	-

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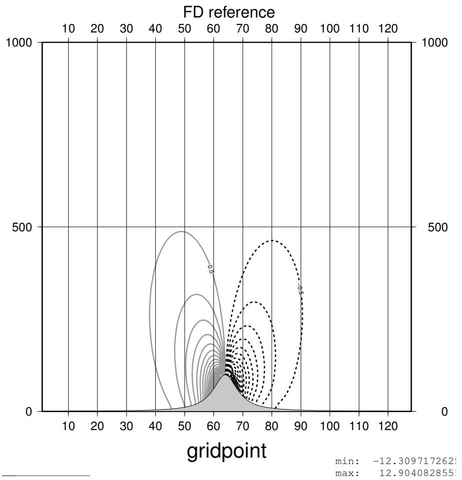
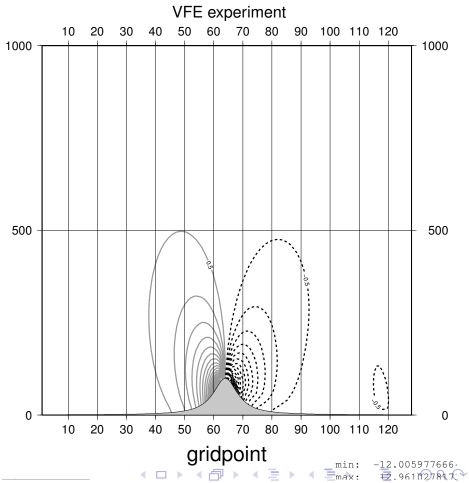
$dz = 20m$, $dx = 20m$, $N = 0.02s^{-1}$, $T_s = 239K$, $V = 15ms^{-1}$, $LEV = 120$, $NDGL = 128$,$H_a = 100m$, $L_a = 1000m$, $dt = 1s$





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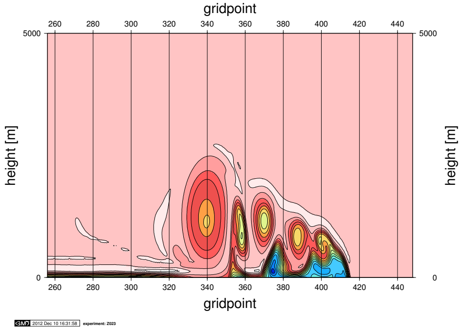
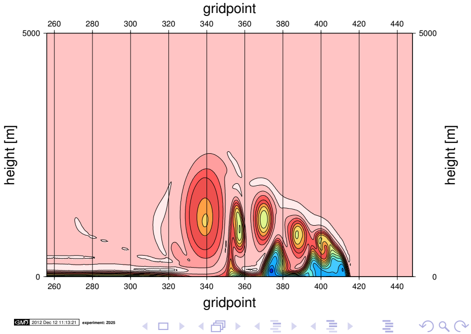
$dz = 50m$, $dx = 50m$, $V = 0ms^{-1}$, $LEV = 200$, $NDGL = 1024$,$dt = 3s$, $\theta = 300K$, elliptic perturbation $\theta' = -15K$ at height $3km$

FD reference



VFE experiment



Flow over Agnesi shaped mountain - linear NH regime

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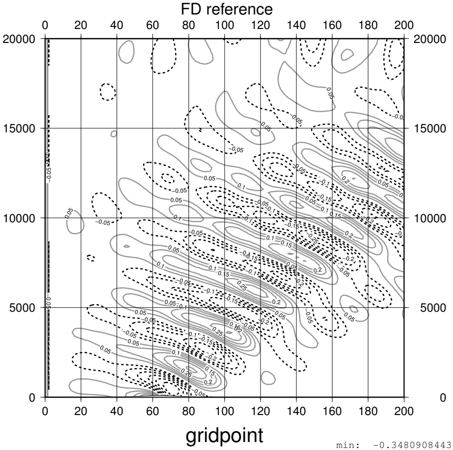
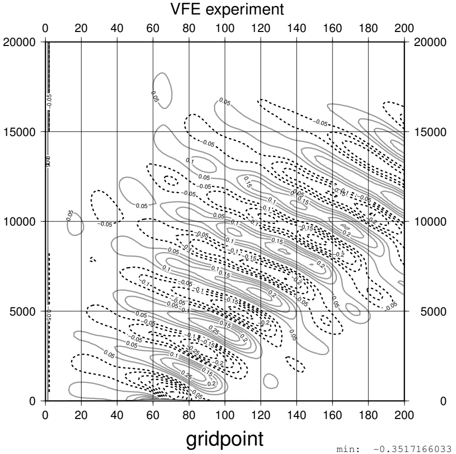
$dz = 200m$, $dx = 80m$, $N = 0.01s^{-1}$, $T_s = 285K$, $V = 4ms^{-1}$, $LEV = 120$, $NDGL = 384$,$H_a = 100m$, $L_a = 1000m$, $dt = 4s$



min: -0.3480908443
max: 0.3259379262



min: -0.3517166033
max: 0.2880283437

Flow over Agnesi shaped mountain - nonlinear NH regime

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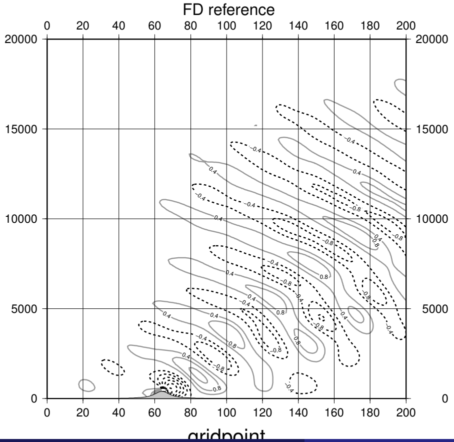
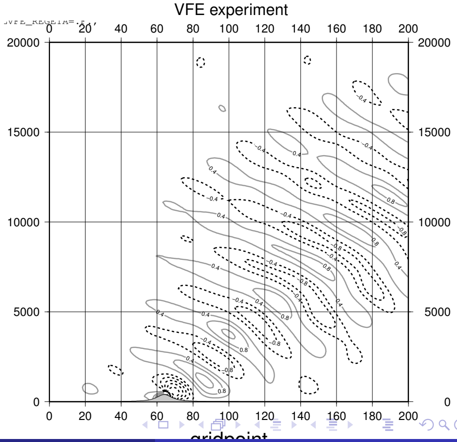
$dz = 200m$, $dx = 80m$, $N = 0.01s^{-1}$, $T_s = 285K$, $V = 4ms^{-1}$, $LEV = 120$, $NDGL = 384$, $H_a = 400m$, $L_a = 400m$, $dt = 4s$





Flow over Agnesi shaped mountain - nonlinear NH regime

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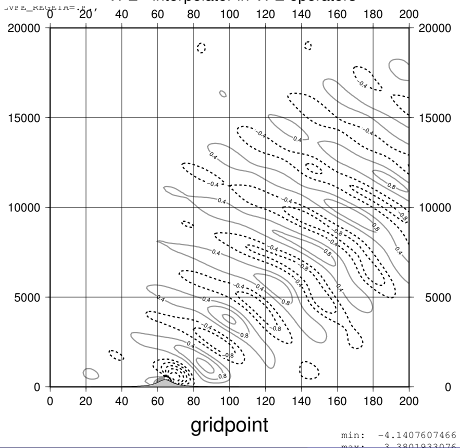
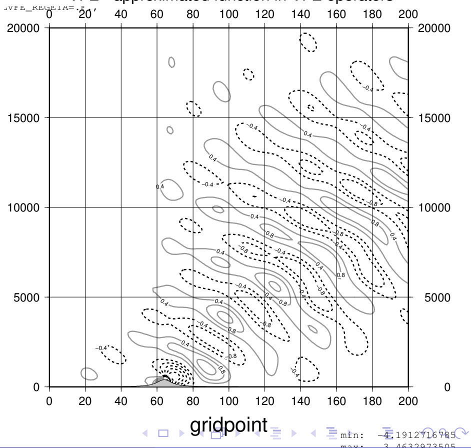
Comparison - interp. scheme vs. approx. scheme





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First adiabatic test over complex terrain

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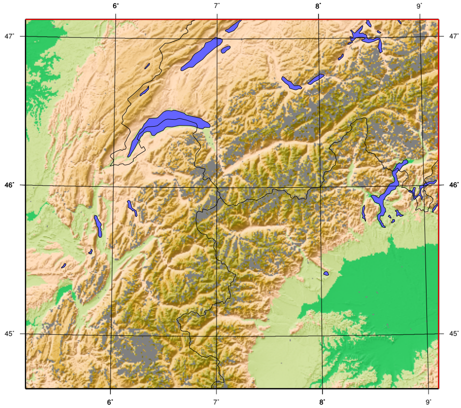
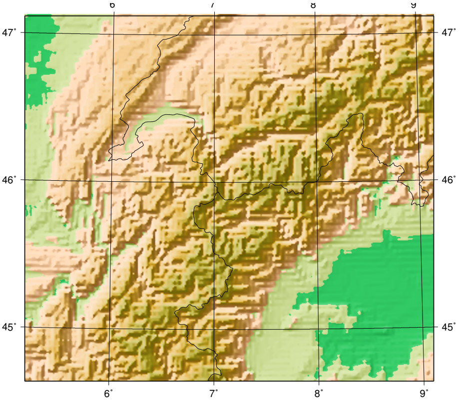
Alpine domain, 28.2.2012 00UTC, 24h forecast

domain - SRTM DEM approx. 100m



domain - E923 orography 1km



ALPS - W cross sections

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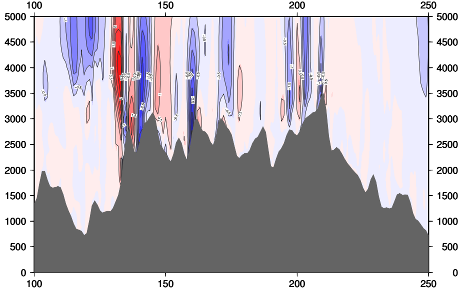
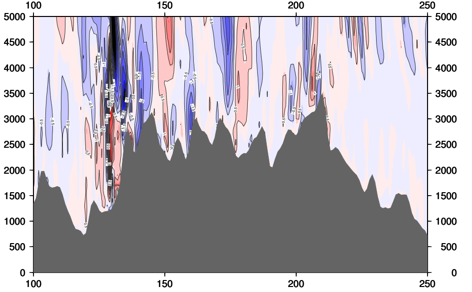
Cross section through the middle of the domain (from west to east)

+24h, $w[ms^{-1}]$, $dt = 30s$



+24h, $w[ms^{-1}]$, $dt = 90s$



Summary

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- The time stepping is stable in linear and non-linear regimes, in 2D and 3D as well,
- The scheme was succesfully tested with 3D adiabatic real cases (more talk of Petra)
- Still to do:
  - put development into new model version (now cy36t1)
  - optimization
  - harmonisation with HY VFE
  - detailed testing of VFE scheme accuracy