Proposal of a modified linear operator for Aladin NH model

Pierre Bénard

CNRM/GMAP/ALGO

1 Introduction

In a recent submitted paper (Bénard, 2003, submitted to MWR) analysing the stability of the EE system in the space-continuous time-discretised context, it appears clearly that the 2-TL EE system with the classical SI time-discretisation is thermally SHB-unstable whatever is the choice of T^*, which was not the case for the HPE system in the same context.
Further examination shows that this dramatic destabilization originates from the fact that the sign of the thermal "non-linear" residuals corresponding to the terms responsible for gravity and acoustic waves systematically have opposite signs. As a consequence, if T^* is chosen so as to stabilize gravity waves, acoustic waves will be unstable, and *vice versa*.
To illustrate this explanation we can examine the couple of following excerpts of the complete linearized EE system :

$$\frac{\partial D}{\partial t} = -R\mathcal{G}\nabla^2 T \tag{1}$$
$$\frac{\partial T}{\partial t} = -\frac{R\overline{T}}{C_v} D \tag{2}$$

$$\frac{\partial \mathsf{d}}{\partial t} = -\frac{g^2}{R\overline{T}}\mathcal{L}\mathcal{P} \tag{3}$$
$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{C_p}{C_v}\mathsf{d} \tag{4}$$

where d is "d_3", $\mathcal{P}$ is "P_2", and other notations are standard to Aladin model.
The first sub-system describes the horizontal propagation of external gravity waves, while the second describes the vertical propagation of acoustic waves. It is clear that these two systems are formally identical the only noticeable difference being the place of the $\overline{T}$ factors (at numerator vs. denominator). For a given set of actual and reference states ($\overline{T}$, T^*) the non-linear residual always have an opposite sign in the two systems. Let us define a non-linearity parameter by $\alpha = (\overline{T} - T^*)/T^*$. The stability properties of the first system for α are thus the same as those of the second system for $-\alpha/(1+\alpha)$. Since the stability of the first system (in 2-TL SI) implies $\alpha < 0$, the stability of the second system implies $\alpha > 0$, and the stability domain of the complete system vanishes.
Several solution could be found to avoid this problem:

- Abuptly remove the SHB problem by using a reference state identical to the current state. However, this requires a complete revisitation of the dynamical core of the model, which is out of the scope of this note. Moreover, this rises many theoretical problems which are not trivially solved.
- Change the set of prognostic variables: the nature of residual terms is highly dependent on the choice of the set of prognostic variables, hence we could expect that a "clever" set removes the corresponding non-linear terms, or at least make them the same sign (maybe for instance the use of θ as a prognostic variable would be a solution). However, such a "clever" set has not been found or examined so far.

Moreover, this would presumably require a big amount of change in the model formulation, namely because the SI scheme would need to be deeply changed.

- Finally, a slight modification of the reference linear operator itself can avoid the problem, by changing the nature of the non-linear residuals. It is this approach which is examined in this note.

2 Background for the SI operator modification

The SI time-discretisation writes symbolically:

$$\frac{\partial X}{\partial t} = \mathcal{M}.X + \mathcal{L}^* \left(\overline{X}^t - X \right) \tag{5}$$

The tradition in NWP is to choose $\mathcal{L}^*$ as the tangent-linear operator of $\mathcal{M}$ around a stationnary reference-state $\mathcal{X}^*$. However, this choice is not strictly a constraint, and any linear operator could be chosen (with the risk that a too "exotic" linear operator will not have stabilizing properties).
For instance, if $\mathcal{L}^*$ is replaced by $(1+\kappa)\mathcal{L}^*$ where κ is a small positive number, we obtain a linear operator which is not the TL of $\mathcal{M}$ around any existing state $\mathcal{X}^*$, and yet we obtain a "decentred SI " scheme which is generally more stable than the normal one.
Here, we explore the impact of choosing a warm reference temperature T^* in the terms responsible for gravity waves, while a cold reference temperature T_a^* is chosen for the terms responsible for acoustic waves. In addition, we note $r = T_a^*/T^*$.
The adiabatic linearized system writes:

$$\frac{\partial D}{\partial t} = -R\mathcal{G}\nabla^2 T + R\overline{T}(\mathcal{G}-\mathcal{I})\nabla^2\mathcal{P} - R\overline{T}\nabla^2 q \tag{6}$$

$$\frac{\partial \mathsf{d}}{\partial t} = -\frac{g^2}{R\overline{T}}\left(1+\sigma\frac{\partial}{\partial\sigma}\right)\left(\sigma\frac{\partial}{\partial\sigma}\right)\mathcal{P} \tag{7}$$

$$\frac{\partial T}{\partial t} = -\frac{R\overline{T}}{C_v}(D+\mathsf{d}) \tag{8}$$

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{S}D - \frac{C_p}{C_v}(D+\mathsf{d}) \tag{9}$$

$$\frac{\partial q}{\partial t} = -\mathcal{N}D, \tag{10}$$

and the reference system writes:

$$\frac{\partial D}{\partial t} = -R\mathcal{G}\nabla^2 T + RT^*(\mathcal{G}-\mathcal{I})\nabla^2\mathcal{P} - RT^*\nabla^2 q \tag{11}$$

$$\frac{\partial \mathsf{d}}{\partial t} = -\frac{g^2}{RT_a^*}\left(1+\sigma\frac{\partial}{\partial\sigma}\right)\left(\sigma\frac{\partial}{\partial\sigma}\right)\mathcal{P} \tag{12}$$

$$\frac{\partial T}{\partial t} = -\frac{RT^*}{C_v}(D+\mathsf{d}) \tag{13}$$

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{S}D - \frac{C_p}{C_v}(D+\mathsf{d}) \tag{14}$$

$$\frac{\partial q}{\partial t} = -\mathcal{N}D, \tag{15}$$

3 Derivation of the Structure Equation

The derivation of the structure equation is important since its allows to demonstrate the feasability of the elimination in the model SI scheme.

The evolution equations for perturbation in the linear system writes:

$$\frac{\partial D}{\partial t} = -R\mathcal{G}^*\Delta T' + gH\mathcal{G}^*\Delta\mathcal{P} - RT^*\Delta\mathcal{P} - RT^*\Delta q \quad (16)$$

$$\frac{\partial T'}{\partial t} = -\frac{RT^*}{C_v}(D + \mathsf{d}) \quad (17)$$

$$\frac{\partial q}{\partial t} = -\mathcal{N}^* D \quad (18)$$

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{S}^* D - \frac{C_p}{C_v}(D + \mathsf{d}) \quad (19)$$

$$\frac{\partial \mathsf{d}}{\partial t} = -\frac{g}{rH}\mathcal{L}^*\mathcal{P} \quad (20)$$

The d equation gives:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\mathcal{L}^*}{rH^2}\right)\mathsf{d} = \frac{\mathcal{L}^*}{rH^2}\left(-gH\mathcal{S}^* + c^2\right)D \quad (21)$$

The time derivative of the divergence equation gives:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)D = \left(-gH\mathcal{G}^* + c^2\right)\Delta\mathsf{d} \quad (22)$$

The elimination can be carried out by multiplying (21) by the left hand side term operator of (22):

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\mathcal{L}^*}{rH^2}\right)\hat{d} &= \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\frac{\mathcal{L}^*}{rH^2}\left(-gH\mathcal{S}^* + c^2\right)D \\ &= \frac{\mathcal{L}^*}{rH^2}\left(-gH\mathcal{S}^* + c^2\right)\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)D \\ &= \frac{\mathcal{L}^*}{rH^2}\left(-gH\mathcal{S}^* + c^2\right)\left(-gH\mathcal{G}^* + c^2\right)\Delta\hat{d}\end{aligned}$$

The latter equation develops in:

$$\frac{\partial^4}{\partial t^4} - c^2\frac{\partial^2}{\partial t^2}\left(\frac{\mathcal{L}^*}{rH^2} + \Delta\right)\hat{d} - \frac{g^2}{r}\mathcal{L}^*\left(\mathcal{S}^*\mathcal{G}^* - \frac{C_p}{C_v}\mathcal{S}^* - \frac{C_p}{C_v}\mathcal{S}^*\right)\Delta\hat{d} = 0$$

An operator $g^2\mathcal{L}^*.\mathcal{A}_2^*$ applying to $\Delta\hat{d}$ appears. This operator simplifies by virtue of the above mentionned properties of the continuous operators (see NH documentation):

$$g^2\mathcal{L}^*.\mathcal{A}_2^*.\Delta\hat{d} = g^2\mathcal{L}^*\left(\mathcal{S}^*\mathcal{G}^* - \frac{C_p}{C_v}\mathcal{S}^* - \frac{C_p}{C_v}\mathcal{G}^*\right)\Delta\hat{d} = N^2c^2\Delta\hat{d}$$

Finally:

$$\left[-\frac{1}{c^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\Delta + \frac{\mathcal{L}^*}{rH^2}\right) + \frac{N^2}{r}\Delta\right]\hat{d} = 0 \quad (23)$$

The derivation is thus formally identical to the normal derivation, simply replacing everywhere $(\mathcal{L}^*/H)$ by $(\mathcal{L}^*/rH)$. Hence we see that the derivation of the structure equation is very similar than for the classical SI formulation. Consequently, the normal modes of the reference system are the same as for the normal SI system (the effect is the same as slightly changing N and H) in (23).

4 Time-Discretisation

The time-discretization is not dramatically changed by this modification, because it relies on the same algebraic manipulations as for the above continuous structure equation. The detail of the time-discretised elimination is not detailed here, but it closely follows the one in the NH documentation.

$$\begin{aligned}
\underline{D'}^{+} - \beta\Delta t\ \Delta'\left[RT^{*}\left(\mathbf{G}^{*}-1\right)\underline{\mathcal{P}}^{+} - R\mathbf{G}^{*}\underline{T}^{+} - RT^{*}\underline{q}^{+}\right] &= \underline{\widetilde{D'^{+}}} \\
\underline{\mathsf{d}}^{+} - \beta\Delta t\ \left(-\frac{g}{rH}\mathbf{L}^{*}\underline{\mathcal{P}}^{+}\right) &= \underline{\widetilde{\mathsf{d}^{+}}} \\
\underline{\mathcal{P}}^{+} - \beta\Delta t\ \left[\left(\mathbf{S}^{*} - \frac{C_p}{C_v}\right)\overline{\mathbf{m}}^{2}\underline{D'}^{+} - \frac{C_p}{C_v}\underline{\mathsf{d}}^{+}\right] &= \underline{\widetilde{\mathcal{P}^{+}}} \\
\underline{T}^{+} - \beta\Delta t\ \left(-\frac{RT^{*}}{C_v}\overline{\mathbf{m}}^{2}\underline{D'}^{+} - \frac{RT^{*}}{C_v}\underline{\mathsf{d}}^{+}\right) &= \underline{\widetilde{T^{+}}} \\
\underline{q}^{+} - \beta\Delta t\ \left(-\mathbf{N}^{*}\overline{\mathbf{m}}^{2}\underline{D'}^{+}\right) &= \underline{\widetilde{q^{+}}}
\end{aligned}$$

with:

$$\begin{aligned}
\underline{\widetilde{D'^{+}}} &= \frac{1}{\mathbf{m}^{2}}\underline{D_E^{+}} + \frac{1}{\mathbf{m}^{2}}\underline{\delta D_{ESI}^{+}} \\
\underline{\widetilde{\mathsf{d}^{+}}} &= \underline{\mathsf{d}_E^{+}} + \underline{\delta\mathsf{d}_{ESI}^{+}} \\
\underline{\widetilde{\mathcal{P}^{+}}} &= \underline{\mathcal{P}_E^{+}} + \underline{\delta\mathcal{P}_{ESI}^{+}} \\
\underline{\widetilde{T^{+}}} &= \underline{T_E^{+}} + \underline{\delta T_{ESI}^{+}} \\
\underline{\widetilde{q^{+}}} &= \underline{q_E^{+}} + \underline{\delta q\ ESI^{+}}
\end{aligned}$$

and:

$$\underline{\delta\mathsf{d}_{ESI}^{+}} = \beta\Delta t\ \left(-\frac{g}{rH}\mathbf{L}^{*}\right)\left(\underline{\mathcal{P}}^{-} - 2\underline{\mathcal{P}}^{0}\right) \tag{24}$$

The only differences with the normal SI set is that a r appears in the LHS of the d equation, and the same r appears in the linear part of the RHS of the d equation, $\underline{\delta\mathsf{d}_{ESI}^{+}}$. The explicit part of equations is of course not touched.

The elimination proceeds as for the normal SI except that the $\mathbf{L}^{*}$ operator is replaced by $(1/r)\mathbf{L}^{*}$. Since the $\mathbf{T}^{*}$ operator is in fact $\mathbf{T}^{*} = \mathbf{L}^{*}.\mathbf{A_2}^{*}$, it also has to be replaced by $(1/r)\mathbf{T}^{*}$.

The final equation to be solve is thus:

$$\underline{\mathsf{d}}^{+} = \mathbf{B}^{-1}.\underline{\mathsf{d}}^{**} \tag{25}$$

with:

$$\mathbf{B} = \left[1 - \beta^{2}\Delta t^{2}c^{2}\overline{\mathbf{m}}^{2}\Delta'\right] - \left[\beta^{2}\Delta t^{2}c^{2}\frac{\mathbf{L}^{*}}{rH^{2}} + \beta^{4}\Delta t^{4}N^{2}c^{2}\overline{\mathbf{m}}^{2}\Delta'\frac{1}{r}\mathbf{T}^{*}\right] \tag{26}$$

and:

$$\underline{\mathsf{d}}^{**} = \left(1 - \beta^{2}\Delta t^{2}c^{2}\overline{\mathbf{m}}^{2}\Delta'\right)\underline{\mathsf{d}}^{*} + \beta^{2}\Delta t^{2}\frac{\mathbf{L}^{*}}{rH^{2}}\left(-RT^{*}\mathbf{S}^{*} + c^{2}\right)\overline{\mathbf{m}}^{2}\underline{D'}^{*} \tag{27}$$

where:

$$\underline{\mathrm{d}}^{*} = \widetilde{\underline{\mathrm{d}^{+}}} + \beta\Delta t\left[\left(-\frac{g}{rH}\mathbf{L}^{*}\right)\widetilde{\underline{\mathcal{P}^{+}}}\right] \quad (28)$$

$$\underline{D'^{*}} = \widetilde{\underline{D'^{+}}} + \beta\Delta t\Delta'\left[RT^{*}\left(\mathbf{G}^{*}-1\right)\widetilde{\underline{\mathcal{P}^{+}}} + \left(-R\mathbf{G}^{*}\right)\widetilde{\underline{T^{+}}} + \left(-RT^{*}\right)\widetilde{\underline{q^{+}}}\right] \quad (29)$$

We see that ihe modification consists in changing the second RHS term of $\underline{\mathrm{d}}^{*}$ in (28), then the second RHS term of $\underline{\mathrm{d}}^{**}$ in (27), then finally, the two terms in the second RHS bracket of $\mathbf{B}$ in (26). In addition, the RHS of (24) has to be modified.

- Modif in (24): `SUBROUTINE GNHDYN` and `LANHSI`: These routine compute the explicit part of the linear computations. Note that in `LANHSI`, two occurences are found, in case of 3-TL or/and 2-TL.
- Modif in (28) : `SUBROUTINE SIDD`: This modifies all computations of $(-g^2/RT)\mathbf{L}^{*}$ by this subroutine, but this subroutine is used with non void $\mathcal{P}$ argument only in the computation of the RHS of (24). Some other used of `SIDD` are made in the code, but always with $\mathcal{P}=0$. It would be more transparent to devote `SIDD` only to the computation of the linear RHS or D, and to replace the d part of `SIDD` by a simple call to `SEVE` with scaling $(-g^2/RT)$.
- Modif in (27) : `SUBROUTINE ESPC`
- Modif in (26) : `SUBROUTINE SUEHEL`

5 Stability analyses

5.1 isothermal framework

In 2-TL SI we saw above that the external gravity waves are stable when $\overline{T} < T^{*}$ (i.e. $\alpha < 0$). The 1D vertical acoustic waves are now stable when $T_a^{*} < \overline{T}$ (i.e. $r-1<\alpha$). As a consequnce, the complete 2-TL SI system can be expected to be stable for $r-1<\alpha<0$, a domain which is not vanishingly small if $r<1$. In the following we chose $r=0.5$ and we examine the stability of various schemes with the classical SHB method for the complete 3D EE system.
The domains for unconditional stability are summarized in the following table:

	Normal (r=1)	modified (r=0.5)
3TL SI	$-0.5<\alpha<1$	$-0.75<\alpha<1$
2TL SI	$\alpha=0$	$-0.5<\alpha<0$
2TL PC	$-0.5<\alpha<1$	$-0.75<\alpha<1$

We see that applying the modified linear operator always contributes to stabilize the time-discretization. For 2-TL SI scheme, the modified linear operator allows a non-vanishing stability domain for the complete 3D system, which is a new feature, never encountered with the classical linear operator.

5.2 non-isothermal framework

The non-isothermal case can be examined in the vertically discretised framework, using the Aladin-NH discretisation. A realistic profile is chosen for $\overline{T}$: surface temperature 285 K, constant lapse-rate in the tropopause, and isothermal stratosphere at 220 K.

In this analysis, T_a^{*} =210K is chosen (colder than $\overline{T}$ at any level), and T^{*} is varied from 220 K to 320 K, and the growth rate is examined for Dt=200s and k=0.0005. The growth rates are depicted on the figure for the normal 2-TL SI scheme with $r=0$ and for the modified SI scheme with T_a^{*} =210 K.

We see that for any T^{*}, the 2-TL SI scheme is unstable, while for warm T^{*}, the modified 2-TL SI becomes stable.

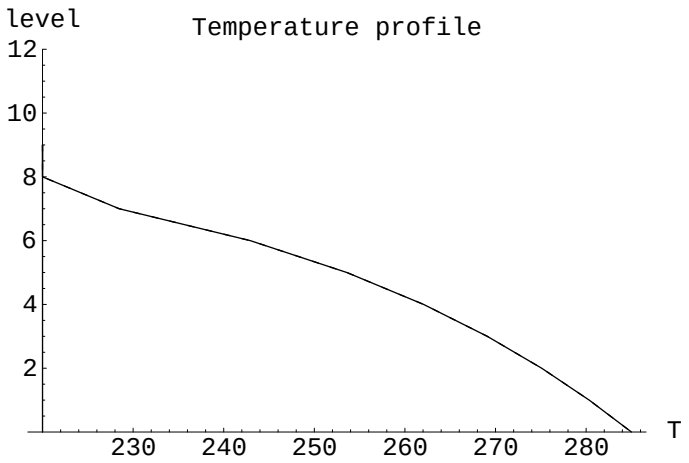

FIG. 1: Thermal profile $\overline{T}$ of the actual atmosphere.

6 Coding aspects

- Introduction of a new SITRA NAMELIST parameter
- Corresponding set-up
- Modification of the matrix $\mathbf{B}$
- Modification of RHS d**

7 Perspectives

It would be nice to discuss a little bit between our small Aladin-NH group to decide if there is something silly in this very preliminary proposal. Any comment or question is welcome of course. Anyway, the coding seems so easy that maybe the best way is to try to code it and think after...

Jean-Francois asks about the loss of accuracy implied by this modification. My first opinion would be : "if r is not chosen too small, the accuracy should be the same as for r=1". In other words, the modification only changes the magnitude of the acoustic non-linear residuals (and inverts their sign). If $r \to 0$ the acoustic residual become unbounded (but with stable sign), and the scheme is not accurate, however if r is chosen such as to invert the sign without generating large magnitude for the non-linear acoustic residual, the level of accuracy should remain basically unchanged. Contrary to decentering, stabilisation would not in this case be obtained at the direct expense of the accuracy.

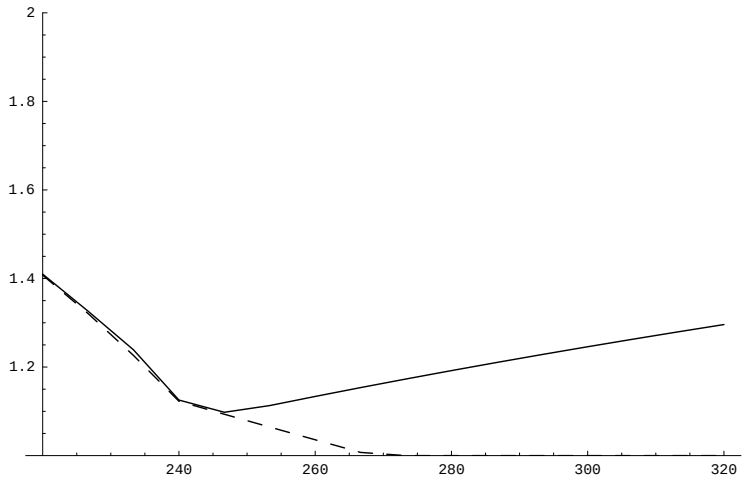

FIG. 2: Asymptotic growth-rates for 1D vertical system as a function of T^*. solid line: original 2-TL SI scheme; dashed line: modified 2-TL SI scheme.