Supporting Information for "Macroscopic modeling of heat and water vapor transfer with phase change in dry snow based on an upscaling method: Influence of air convection"

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Contents of this file: This supporting information provides the details of the asymptotic analysis to derive the macroscopic modeling in the Case B1 (diffusion and moderate convection + source terms) and the Case C1 (strong convection + dispersion + source terms), as referred in the main article.

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Appendix: Asymptotic analysis - Case B1

Taking into account the order of magnitude of the dimensionless numbers in the Case B1, $\left[\mathrm{F_i^T}\right] = \mathcal{O}\left(\left[\mathrm{F_a^T}\right]\right) = \mathcal{O}\left(\left[\mathrm{F_a^\rho}\right]\right) = \mathcal{O}(\varepsilon^2)$, $[\mathrm{Re}] = \mathcal{O}\left(\left[\mathrm{Pe^T}\right]\right) = \mathcal{O}\left(\left[\mathrm{Pe^\rho}\right]\right) = \mathcal{O}(\varepsilon)$, $[\mathrm{Q}] = \mathcal{O}(\varepsilon^{-1})$, $[\mathrm{N}] = \mathcal{O}(\varepsilon^{-1})$, $[\mathrm{M}] = \mathcal{O}(\varepsilon^2)$, $[\mathrm{K}] = \mathcal{O}(1)$, $[\mathrm{H}] = \mathcal{O}(\varepsilon^2)$, $[\mathrm{W}] = \mathcal{O}(\varepsilon^2)$, the dimensionless microscopic description (Equations (13)-(22)) becomes:

$$\varepsilon \rho_a^* \mathbf{v_a}^* \cdot \mathbf{grad}^* \, \mathbf{v_a}^* = \mu_a^* \Delta^* \mathbf{v_a}^* - \varepsilon^{-1} \mathbf{grad}^* \, p_a^* \quad \text{in } \Omega_a \tag{B.1}$$

$$\mathrm{div}^* \, \mathbf{v_a}^* = 0 \quad \text{in } \Omega_a \tag{B.2}$$

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^*(\kappa_i^* \mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{B.3}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} + \varepsilon \rho_a^* C_a^* \mathbf{v_a}^* \cdot \mathbf{grad}^* \, T_a^* - \mathrm{div}^*(\kappa_a^* \mathbf{grad}^* T_a^*) = 0 \quad \text{in } \Omega_a \tag{B.4}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} + \varepsilon \mathbf{v_a}^* \cdot \mathbf{grad}^* \, \rho_v^* - \mathrm{div}^*(D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{B.5}$$

$$\rho_a^* (\varepsilon^2 \mathbf{w}^* - \mathbf{v_a}^*) \cdot \mathbf{n^\Gamma} = -\varepsilon \rho_i^* \mathbf{w}^* \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma \tag{B.6}$$

$$\mathbf{v_a}^* \cdot \mathbf{t^\Gamma} = 0 \quad \text{on } \Gamma \tag{B.7}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{B.8}$$

$$\kappa_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n^\Gamma} - \kappa_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n^\Gamma} = \varepsilon^2 L_{sg}^* \mathbf{w}^* \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma \tag{B.9}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n^\Gamma} = \varepsilon^2 \rho_i^* \mathbf{w}^* \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma. \tag{B.10}$$

This set of equations is completed by the Hertz-Knudsen equation and the Clausius Clapeyron's law, Eq. (11) and (12), expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n^\Gamma} = \frac{1}{\beta^*} \left[\frac{\rho_v^* - \rho_{vs}^*(T_a^*)}{\rho_{vs}^*(T_a^*)} - d_0^* K^* \right] \quad \text{on } \Gamma \tag{B.11}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp \left[\frac{L_{sg}^* m^*}{\rho_i^* k^*} \left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*} \right) \right] \tag{B.12}$$

Note that the steady air flow equation is here written using the Laplacien symbol to shorten the equations length

Fluid flow

Introducing asymptotic expansions for $\mathbf{v}_a^*$ and p_a^* in the relations (B.1) gives at the lowest order ε^{-1}:

$$\mathbf{grad}_{y^*}\, p_a^{*(0)} = 0 \quad \text{in } \Omega_a \tag{B.13}$$

where the unknown $p_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*)$ is $\mathbf{y}^*$-periodic. It can be shown [*Auriault et al.*, 2009] that this relation implies that:

$$p_a^{*(0)} = p_a^{*(0)}(\mathbf{x}^*). \tag{B.14}$$

At the first order, the pressure is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. constant over a period or REV. Taking into account these results, equations (B.1, B.2, B.7, B.6) of order ε^0 give the following second-order problem:

$$\mu_a^* \Delta_{y^*} \mathbf{v_a}^{*(0)} - \mathbf{grad}_{y^*}\, p_a^{*(1)} - \mathbf{grad}_{x^*}\, p_a^{*(0)} = 0 \quad \text{in } \Omega_a \tag{B.15}$$

$$\mathrm{div}_{y^*}\, \mathbf{v_a}^{*(0)} = 0 \quad \text{in } \Omega_a \tag{B.16}$$

$$\mathbf{v_a}^{*(0)} \cdot \mathbf{t}^{\boldsymbol{\Gamma}} = 0 \quad \text{on } \Gamma \tag{B.17}$$

$$\mathbf{v_a}^{*(0)} \cdot \mathbf{n}^{\boldsymbol{\Gamma}} = 0 \quad \text{on } \Gamma \tag{B.18}$$

where $\mathbf{v}_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$ and $p_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$ are the $\mathbf{y}^*$-periodic unknowns, which represent respectively the fluid velocity and the pressure fluctuation in a REV induced by a given macroscopic gradient of pressure $\mathbf{grad}_{x^*} p_a^{*(0)}$. It can be shown that $\mathbf{v_a}^{*(0)}$ and $p_a^{*(1)}$ are linear functions of $\mathbf{grad}_{x^*} p_a^{*(0)}$, and that $p_a^{*(1)}$ is expressed modulo an arbitrary function $\widetilde{p}_a^{*(1)}(\mathbf{x}^*)$ [*Auriault et al.*, 2009]:

$$p_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{b}^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} p_a^{*(0)} + \widetilde{p}_a^{*(1)}(\mathbf{x}^*) \tag{B.19}$$

$$\mathbf{v}_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{k}^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} p_a^{*(0)} \tag{B.20}$$

where $\mathbf{k}^*(\mathbf{y}^*)$ is a second order tensor and $\mathbf{b}^*(\mathbf{y}^*)$ is a y^*-periodic vector and average zero over the REV, $\langle \mathbf{b}^* \rangle = \mathbf{0}$. This latter condition ensures the uniqueness of $\mathbf{b}^*$. $\mathbf{b}^*$ characterizes the fluctuation of pressure at the pore scale induced by the macroscopic gradient. Introducing (B.19) and (B.20) in the set (B.15)-(B.17), the tensor $\mathbf{k}^*(\mathbf{y}^*)$ and vector $\mathbf{b}^*(\mathbf{y}^*)$ are solution of the following boundary value problem in a compact form:

$$\mu_a^* \Delta_{y^*} \mathbf{k}^* - \mathbf{grad}_{y^*} \mathbf{b}^* - \mathbf{I} = \mathbf{0} \quad \text{in } \Omega_a \tag{B.21}$$

$$\mathbf{grad}_{y^*} \mathbf{k}^* = \mathbf{0} \quad \text{in } \Omega_a \tag{B.22}$$

$$\mathbf{k}^* = \mathbf{0} \quad \text{on } \Gamma \tag{B.23}$$

At the next order, equations (B.2, B.7, B.6) are written:

$$\mathrm{div}_{x^*} \, \mathbf{v_a}^{*(0)} + \mathrm{div}_{y^*} \, \mathbf{v_a}^{*(1)} = 0 \quad \text{in } \Omega_a \tag{B.24}$$

$$\mathbf{v_a}^{*(1)} \cdot \mathbf{t^\Gamma} = 0 \quad \text{on } \Gamma \tag{B.25}$$

$$\rho_a^* \mathbf{v_a}^{*(1)} \cdot \mathbf{n^\Gamma} = -\rho_i^* \mathbf{w}^{*(0)} \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma \tag{B.26}$$

where the unknown $\mathbf{v}_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$ is $\mathbf{y}^*$-periodic. Integrating equation (B.24) over Ω_a and then using the divergence theorem, boundary condition (B.26) and the periodicity condition, the dimensionless macroscopic mass balance takes the form

$$\mathrm{div}_{x^*}(\langle \mathbf{v}_a^{*(0)} \rangle) + \left(1 - \frac{\rho_i^*}{\rho_a^*}\right) \mathrm{SSA}_V \; w_n^{*(0)} = 0 \tag{B.27}$$

where

$$\langle \mathbf{v}_a^{*(0)} \rangle = -\frac{\mathbf{K}^{\mathrm{eff}*}}{\mu_a^*} \mathbf{grad}_{x^*} \; p_a^{*(0)} \tag{B.28}$$

$$\mathbf{K}^{\mathrm{eff}*} = \frac{1}{|\Omega|} \int_{\Omega_a} \mathbf{k} \, \mathrm{d} \, \Omega. \tag{B.29}$$

and $w_n^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is the normal interface velocity at the zero order due to the phase change given by the Hertz-Knudsen equation (B.67) and the Clausius Clapeyron's law (B.66).

Heat transfer

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (B.3, B.4, B.8, B.9) gives at the lowest order ε^0:

$$\mathrm{div}_{y^*}(\kappa_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{B.30}$$

$$\mathrm{div}_{y^*}(\kappa_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{B.31}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{B.32}$$

$$(\kappa_i^* \mathbf{grad}_{y^*} T_i^{*(0)} - \kappa_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{B.33}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. It can be shown [*Auriault et al.*, 2009] that the obvious solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{B.34}$$

At the first order, the temperature is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field. Taking into account these results, equations (B.3, B.4, B.8, B.9) of order ε give the following second-order problem:

$$\mathrm{div}_{y^*}(\kappa_i^* (\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{B.35}$$

$$\mathrm{div}_{y^*}(\kappa_a^* (\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{B.36}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{B.37}$$

$$(\kappa_i^* (\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - \kappa_a^* (\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{B.38}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*, t)$ [*Auriault et al.*, 2009]:

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_i^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{B.39}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{t}_a^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{B.40}$$

where $\mathbf{t}_i^*(\mathbf{y}^*)$ and $\mathbf{t}_a^*(\mathbf{y}^*)$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. Introducing (B.39) and (B.40) in the set (B.35)-(B.38), these two vectors are solution of the following boundary value problem in a compact form:

$$\text{div}_{y^*}(\kappa_i^*(\mathbf{grad}_{y^*} \mathbf{t}_i^* + \mathbf{I})) = 0 \quad \text{in } \Omega_i \tag{B.41}$$

$$\text{div}_{y^*}(\kappa_a^*(\mathbf{grad}_{y^*} \mathbf{t}_a^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{B.42}$$

$$\mathbf{t}_i^* = \mathbf{t}_a^* \quad \text{on } \Gamma \tag{B.43}$$

$$(\kappa_i^*(\mathbf{grad}_{y^*} \mathbf{t}_i^* + \mathbf{I}) - \kappa_a^*(\mathbf{grad}_{y^*} \mathbf{t}_a^* + \mathbf{I})) \cdot \mathbf{n}^{\boldsymbol{\Gamma}} = 0 \quad \text{on } \Gamma \tag{B.44}$$

$$\frac{1}{|\Omega|} \int_{\Omega} (\mathbf{t}_{\mathrm{a}}^* + \mathbf{t}_{\mathrm{i}}^*) \mathrm{d}\Omega = \mathbf{0} \tag{B.45}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the equations (B.3, B.4, B.8, B.9) of order ε^2:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \text{div}_{y^*}(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}))$$

$$-\text{div}_{x^*}(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{B.46}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} + \rho_a^* C_a^* \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{x^*} T^{*(0)} + \mathbf{grad}_{y^*} T_a^{*(1)})$$

$$-\mathrm{div}_{y^*}(\kappa_a^*(\mathbf{grad}_{y^*}T_a^{*(2)}+\mathbf{grad}_{x^*}T_a^{*(1)}))-\mathrm{div}_{x^*}(\kappa_a^*(\mathbf{grad}_{y^*}T_a^{*(1)}+\mathbf{grad}_{x^*}T^{*(0)}))=0 \quad \text{in } \Omega_a \tag{B.47}$$

$$T_i^{*(2)}=T_a^{*(2)} \quad \text{on } \Gamma \tag{B.48}$$

$$(\kappa_i^*(\mathbf{grad}_{y^*}T_i^{*(2)}+\mathbf{grad}_{x^*}T_i^{*(1)})-\kappa_a^*(\mathbf{grad}_{y^*}T_a^{*(2)}+\mathbf{grad}_{x^*}T_a^{*(1)}))\cdot\mathbf{n}^{\mathbf{\Gamma}}=L_{sg}^*w_n^{*(0)} \quad \text{on } \Gamma \tag{B.49}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*,\mathbf{y}^*,t)$ and $T_a^{*(2)}(\mathbf{x}^*,\mathbf{y}^*,t)$ are $\mathbf{y}^*$-periodic, $\mathbf{v}_a^{*(0)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is given by the equation (B.20) and verifies the relations (B.16) – (B.18), and $w_n^{*(0)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is the normal interface velocity at the zero order due to the phase change given by the Hertz-Knudsen (B.67) and the Clausius Clapeyron's law (B.66). Consequently, integrating (B.46) over Ω_i and (B.47) over Ω_a, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B.49) lead to the first order dimensionless macroscopic description for the heat transfer:

$$(\rho C)^{\mathrm{eff}*}\frac{\partial T^{*(0)}}{\partial t^*}+\rho_a^*C_a^*\langle\mathbf{v}_a^{*(0)}\rangle\cdot\mathbf{grad}_{x^*}T^{*(0)}-\mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{eff}*}\mathbf{grad}_{x^*}\ T^{*(0)})=\mathrm{SSA}_V L_{sg}^*w_n^{*(0)} \tag{B.50}$$

where $\mathrm{SSA_V}=|\Gamma|/|\Omega|$ is the specific surface area, $(\rho C)^{\mathrm{eff}*}$ and $\mathbf{k}^{\mathrm{eff}*}$ are the dimensionless effective thermal capacity and the effective dimensionless conductivity respectively, defined as:

$$(\rho C)^{\mathrm{eff}*}=(1-\phi)\rho_i^*C_i^*+\phi\rho_a^*C_a^* \tag{B.51}$$

$$\mathbf{k}^{\mathrm{eff}*}=\frac{1}{|\Omega|}\left(\int_{\Omega_\mathrm{a}}\kappa_a^*(\mathbf{grad}_{y^*}\mathbf{t}_a^*(\mathbf{y}^*)+\mathbf{I})\mathrm{d}\Omega+\int_{\Omega_\mathrm{i}}\kappa_i^*(\mathbf{grad}_{y^*}\mathbf{t}_i^*(\mathbf{y}^*)+\mathbf{I})\mathrm{d}\Omega\right) \tag{B.52}$$

where ϕ is the porosity.

Water vapor transfer

Introducing asymptotic expansions for ρ_v^* in the relations (B.5, B.10) gives at the lowest order (ε^0)

$$\mathrm{div}_{y^*}(D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{B.53}$$

$$D_v^* \mathbf{grad}_{y^*} \rho_v^{*(0)} \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma. \tag{B.54}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. It can be shown [*Auriault et al.*, 2009] that the solution of the above boundary value problem is given by:

$$\rho_v^{*(0)} = \rho_v^{*(0)}(\mathbf{x}^*, t). \tag{B.55}$$

At the first order, the water vapor density is independent of the microscopic dimensionless variable $\mathbf{y}^*$. Taking into account these results, the second-order problem is given by the equations (B.5, B.10) of order ε:

$$\mathrm{div}_{y^*}(D_v^* (\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{B.56}$$

$$D_v^* (\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma. \tag{B.57}$$

where the unknown $\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} \rho_v^{*(0)}$ is given. The solution of the above boundary value problem appears as a linear function of the macroscopic gradient, modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t)$ [*Auriault et al.*, 2009]:

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{g}_v^*(\mathbf{y}^*) \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)} + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t) \tag{B.58}$$

where $\mathbf{g}_v^*(\mathbf{y}^*)$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale. Introducing (B.58) in the set (B.56)-(B.57), this vector is solution of the following boundary value problem in a compact form:

$$\mathrm{div}_{y^*}(D_v^* (\mathbf{grad}_{y^*} \mathbf{g}_v^* + \mathbf{I})) = 0 \quad \text{in } \Omega_a \tag{B.59}$$

$$D_v^* (\mathbf{grad}_{y^*} \mathbf{g}_v^* + \mathbf{I}) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{B.60}$$

$$\frac{1}{|\Omega|}\int_{\Omega_a} \mathbf{g}_v^* \mathrm{d}\Omega = \mathbf{0} \tag{B.61}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the equations (B.5, B.10) of order ε^2:

$$\frac{\partial \rho_v^{*(0)}}{\partial t^*} + \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{x^*}\rho_v^{*(0)} + \mathbf{grad}_{y^*}\rho_v^{*(1)}) - \mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(2)}))$$

$$-\mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)} + \mathbf{grad}_{x^*}\rho_v^{*(1)})) = 0 \quad \text{in } \Omega_i \tag{B.62}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(2)} + \mathbf{grad}_{x^*}\rho_v^{*(2)}) \cdot \mathbf{n}^{\Gamma} = \rho_i^* w_n^{*(0)} \quad \text{on } \Gamma \tag{B.63}$$

where the unknown $\rho_v^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic, $\mathbf{v}_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is given by the equation (B.20) (and verifies the relations (B.16) and (B.18)) and and $w_n^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is the normal interface velocity due to the phase change at the zero order given by the Hertz-Knudsen equation (B.67) and the Clausius Clapeyron's law (B.66). Integrating (B.62) over Ω_a, and then using the divergence theorem, the periodicity condition, and the boundary conditions (B.63) lead to the first order dimensionless macroscopic description for the water vapor transfer:

$$\phi\frac{\partial \rho_v^{*(0)}}{\partial t} + \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*}\rho_v^{*(0)} - \mathrm{div}_{x^*}(\mathbf{D}^{*\mathrm{eff}}\mathbf{grad}_{x^*}\rho_v^{*(0)}) = -\mathrm{SSA}_V \rho_i^* w_n^{*(0)} \tag{B.64}$$

where $\mathrm{SSA_V} = |\Gamma|/|\Omega|$ is the surface area and $\mathbf{D}^{*\mathrm{eff}}$ is the dimensionless effective diffusion tensor defined as:

$$\mathbf{D}^{\mathrm{eff}*} = \frac{1}{|\Omega|}\int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{g}_v^*(\mathbf{y}^*) + \mathbf{I})\mathrm{d}\Omega \tag{B.65}$$

Expression of $w_n^{*(0)}$

The asymptotic analysis for the the Clausius Clapeyron's law and the Hertz-Knudsen equation are presented in *Calonne et al.* [2014]. They obtained

$$\rho_{vs}^{*(0)}(T^{*(0)}) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp\left[\frac{L_{sg}^* m^*}{\rho_i^* k^*}\left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T^{*(0)}}\right)\right] \tag{B.66}$$

$$w_n^{*(0)} = \frac{1}{\beta^*} \left[\frac{\rho_v^{*(0)} - \rho_{vs}^{*(0)}(T^{*(0)})}{\rho_{vs}^{*(0)}(T^{*(0)})} - d_0^* K^* \right] \tag{B.67}$$

The relations (B.67) and (B.66) show that the normal velocity $w_n^{*(0)}$ arising in the boundary conditions (B.49) and (B.63) does not depend on $\mathbf{y}^*$.

Appendix: Asymptotic analysis - Case C1

Taking into account the order of magnitude of the dimensionless numbers in the Case C1, $\left[\mathrm{F_i^T}\right] = \mathcal{O}\left(\left[\mathrm{F_a^T}\right]\right) = \mathcal{O}\left([\mathrm{F_a^\rho}]\right) = \mathcal{O}(\varepsilon)$, $[\mathrm{Re}] = \mathcal{O}\left(\left[\mathrm{Pe^T}\right]\right) = \mathcal{O}\left([\mathrm{Pe^\rho}]\right) = \mathcal{O}\left(1\right)$, $[\mathrm{Q}] = \mathcal{O}\left(\varepsilon^{-1}\right)$, $[\mathrm{N}] = \mathcal{O}\left(\varepsilon^{-1}\right)$, $[\mathrm{M}] = \mathcal{O}\left(\varepsilon^{3}\right)$, $[\mathrm{K}] = \mathcal{O}\left(1\right), [\mathrm{H}] = \mathcal{O}\left(\varepsilon^{2}\right), [\mathrm{W}] = \mathcal{O}\left(\varepsilon^{2}\right)$, the dimensionless microscopic description (Equations (13)-(22)) becomes:

$$\rho_a^* \mathbf{v_a}^* \cdot \mathbf{grad}^* \, \mathbf{v_a}^* = \mu_a^* \Delta^* \mathbf{v_a}^* - \varepsilon^{-1} \mathbf{grad}^* \, p_a^* \quad \text{in } \Omega_a \tag{C.1}$$

$$\mathrm{div}^* \, \mathbf{v_a}^* = 0 \quad \text{in } \Omega_a \tag{C.2}$$

$$\varepsilon^2 \rho_i^* C_i^* \frac{\partial T_i^*}{\partial t^*} - \mathrm{div}^* (\kappa_i^* \mathbf{grad}^* T_i^*) = 0 \quad \text{in } \Omega_i \tag{C.3}$$

$$\varepsilon^2 \rho_a^* C_a^* \frac{\partial T_a^*}{\partial t^*} + \rho_a^* C_a^* \mathbf{v_a}^* \cdot \mathbf{grad}^* \, T_a^* - \mathrm{div}^* (\kappa_a^* \mathbf{grad}^* T_a^*) = 0 \quad \text{in } \Omega_a \tag{C.4}$$

$$\varepsilon^2 \frac{\partial \rho_v^*}{\partial t^*} + \mathbf{v_a}^* \cdot \mathbf{grad}^* \, \rho_v^* - \mathrm{div}^* (D_v^* \mathbf{grad}^* \rho_v^*) = 0 \quad \text{in } \Omega_a \tag{C.5}$$

$$\rho_a^* (\varepsilon^2 \mathbf{w}^* - \mathbf{v_a}^*) \cdot \mathbf{n^\Gamma} = -\varepsilon \rho_i^* \mathbf{w}^* \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma \tag{C.6}$$

$$\mathbf{v_a}^* \cdot \mathbf{t^\Gamma} = 0 \quad \text{on } \Gamma \tag{C.7}$$

$$T_i^* = T_a^* \quad \text{on } \Gamma \tag{C.8}$$

$$\kappa_i^* \mathbf{grad}^* T_i^* \cdot \mathbf{n^\Gamma} - \kappa_a^* \mathbf{grad}^* T_a^* \cdot \mathbf{n^\Gamma} = \varepsilon^2 L_{sg}^* \mathbf{w}^* \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma \tag{C.9}$$

$$D_v^* \mathbf{grad}^* \rho_v^* \cdot \mathbf{n^\Gamma} = \varepsilon^2 \rho_i^* \mathbf{w}^* \cdot \mathbf{n^\Gamma} \quad \text{on } \Gamma. \tag{C.10}$$

This set of equations is completed by the Hertz-Knudsen equation and the Clausius Clapeyron's law, Eq. (11) and (12), expressed in dimensionless form as:

$$w_n^* = \mathbf{w}^* \cdot \mathbf{n^\Gamma} = \frac{1}{\beta^*} \left[\frac{\rho_v^* - \rho_{vs}^*(T_a^*)}{\rho_{vs}^*(T_a^*)} - d_0^* K^* \right] \quad \text{on } \Gamma \tag{C.11}$$

$$\rho_{vs}^*(T_a^*) = \rho_{vs}^{\mathrm{ref}*}(T^{\mathrm{ref}*}) \exp \left[\frac{L_{sg}^* m^*}{\rho_i^* k^*} \left(\frac{1}{T^{\mathrm{ref}*}} - \frac{1}{T_a^*} \right) \right] \tag{C.12}$$

Here again, note that the steady air flow equation is written using the Laplacien symbol to shorten the equations length

Fluid flow

As in the case B1, introducing asymptotic expansions for $\mathbf{v}_a^*$ and p_a^* in the relations (C.1) gives at the lowest order ε^{-1}:

$$\mathbf{grad}_{y^*}\, p_a^{*(0)} = 0 \quad \text{in } \Omega_a, \tag{C.13}$$

where the unknown $p_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*)$ is $\mathbf{y}^*$-periodic. It can be shown [*Auriault et al.*, 2009] that this relation implies that:

$$p_a^{*(0)} = p_a^{*(0)}(\mathbf{x}^*). \tag{C.14}$$

At the first order, the pressure is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. constant over a period or REV. Taking into account these results, equations (C.1, C.2, C.7, C.6) of order ε^0 give now the following second-order problem:

$$\mu_a^* \Delta_{y^*} \mathbf{v_a}^{*(0)} - \mathbf{grad}_{y^*}\, p_a^{*(1)} - \mathbf{grad}_{x^*}\, p_a^{*(0)} = \rho_a^* \mathbf{v_a}^{*(0)}\, \mathbf{grad}_{y^*}^*\, \mathbf{v_a}^{*(0)} \quad \text{in } \Omega_a \tag{C.15}$$

$$\text{div}_{y^*}\, \mathbf{v_a}^{*(0)} = 0 \quad \text{in } \Omega_a \tag{C.16}$$

$$\mathbf{v_a}^{*(0)} \cdot \mathbf{t}^{\boldsymbol{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.17}$$

$$\mathbf{v_a}^{*(0)} \cdot \mathbf{n}^{\boldsymbol{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.18}$$

where $\mathbf{v}_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$ and $p_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$ are the $\mathbf{y}^*$-periodic unknowns. By contrast to the Case B1, the equation (C.15) is strongly nonlinear. Consequently, $\mathbf{v_a}^{*(0)}$ appears as a nonlinear function $\mathbf{f}$ of the macroscopic pressure gradient $\mathbf{grad}_{x^*} p_a^{*(0)}$, of $\mathbf{y}^*$ and of the fluid properties (ρ_a^* μ_a^*), (see [*Auriault et al.*, 2009] and references herein for more details):

$$\mathbf{v}_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*) = -\mathbf{f}(\mathbf{grad}_{x^*} p_a^{*(0)}, \mathbf{y}^*, \rho_a^*, \mu_a^*) \tag{C.19}$$

A similar relation stands for the pressure $p_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$. At the next order, equations (C.2, C.7, C.6) are written:

$$\text{div}_{x^*}\, \mathbf{v_a}^{*(0)} + \text{div}_{y^*}\, \mathbf{v_a}^{*(1)} = 0 \quad \text{in } \Omega_a \tag{C.20}$$

$$\mathbf{v_a}^{*(1)} \cdot \mathbf{t}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.21}$$

$$\mathbf{v_a}^{*(1)} \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.22}$$

where the unknown $\mathbf{v}_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*)$ is $\mathbf{y}^*$-periodic. Let us remark that the equation (C.22) does not present a left term as in the case B1, since now $[\mathrm{M}] = \mathcal{O}(\varepsilon^3)$. As in the Case B1, integrating equation (C.20) over Ω_a and then using the divergence theorem, boundary condition (C.22) and the periodicity condition, the dimensionless macroscopic mass balance takes the form

$$\mathrm{div}_{x^*}(\langle \mathbf{v}_a^{*(0)} \rangle) = 0 \tag{C.23}$$

where the dimensionless macroscopic flow law is written:

$$\langle \mathbf{v}_a^{*(0)} \rangle = -\frac{1}{|\Omega|} \int_{\Omega_a} \mathbf{f}(\mathbf{grad}_{x^*} p_a^{*(0)}, \mathbf{y}^*, \rho_a^*, \mu_a^*), \mathrm{d}\Omega = -\mathbf{F}(\mathbf{grad}_{x^*} p_a^{*(0)}, \text{microstructure}, \rho_a^*, \mu_a^*) \tag{C.24}$$

Heat transfer

Introducing asymptotic expansions for T_i^* and T_a^* in the relations (C.3, C.4, C.8, C.9) gives at the lowest order ε^0:

$$\mathrm{div}_{y^*}(\kappa_i^* \mathbf{grad}_{y^*} T_i^{*(0)}) = 0 \quad \text{in } \Omega_i \tag{C.25}$$

$$\rho_a^* C_a^* \mathbf{v}_a^{*(0)} \cdot \mathbf{grad}_{y^*} T^{*(0)} - \mathrm{div}_{y^*}(\kappa_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) = 0 \quad \text{in } \Omega_a \tag{C.26}$$

$$T_i^{*(0)} = T_a^{*(0)} \quad \text{on } \Gamma \tag{C.27}$$

$$(\kappa_i^* \mathbf{grad}_{y^*} T_i^{*(0)} - \kappa_a^* \mathbf{grad}_{y^*} T_a^{*(0)}) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.28}$$

where the unknowns $T_i^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. It can be shown [*Auriault et al.*, 2009; *Geindreau and Auriault*, 2001] that the solution of the above boundary value problem is given by:

$$T_i^{*(0)} = T_a^{*(0)} = T^{*(0)}(\mathbf{x}^*, t). \tag{C.29}$$

At the first order, the temperature is independent of the microscopic dimensionless variable $\mathbf{y}^*$, i.e. we have only one temperature field. Taking into account these results, equations (C.3, C.4, C.8, C.9), of order ε give the following second-order problem:

$$\rho_i^* C_i^* \frac{\partial T^{*(0)}}{\partial t^*} - \mathrm{div}_{y^*}(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C.30}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(0)}}{\partial t^*} + \rho_a^* C_a^* \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{x^*} T^{*(0)} + \mathbf{grad}_{y^*} T_a^{*(1)})$$
$$-\mathrm{div}_{y^*}(\kappa_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C.31}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{C.32}$$

$$(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - \kappa_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.33}$$

where the unknowns $T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*} T^{*(0)}$ is given. $\mathbf{v}_a^{*(0)}$ is given by the relation (C.19). Integrating (C.30) over Ω_i and (C.31) over Ω_a and taking into account the condition of periodicity, the boundary condition (C.33) and the relation (C.19), we obtain the following first order macroscopic description:

$$(\rho C)^{\mathrm{eff}*} \frac{\partial T^{*(0)}}{\partial t^*} + \rho_a^* C_a^* \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} T^{*(0)} = 0 \tag{C.34}$$

where $\langle \mathbf{v}_a^{*(0)} \rangle$ and $(\rho C)^{\mathrm{eff}*}$ are given by the relations (C.24) and (B.51) respectively. As expected, the convection alone is present at the first order of approximation.

The first correction of this macroscopic model will bring the diffusion into play. Using the relation (C.34), the boundary value problem can be put in the form:

$$-\beta^* \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} T^{*(0)} - \mathrm{div}_{y^*}(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C.35}$$

$$-\gamma^* \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} T^{*(0)} + \rho_a^* C_a^* \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{x^*} T^{*(0)} + \mathbf{grad}_{y^*} T_a^{*(1)})$$
$$-\mathrm{div}_{y^*}(\kappa_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C.36}$$

$$T_i^{*(1)} = T_a^{*(1)} \quad \text{on } \Gamma \tag{C.37}$$

$$(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)}) - \kappa_a^*(\mathbf{grad}_{y^*} T_a^{*(1)} + \mathbf{grad}_{x^*} T^{*(0)})) \cdot \mathbf{n}^{\mathbf{\Gamma}} = 0 \quad \text{on } \Gamma \tag{C.38}$$

where $\beta^* = (\rho_a^* C_a^*)(\rho_i^* C_i^*)/(\rho C)^{\text{eff}*}$ and $\gamma^* = (\rho_a^* C_a^*)^2/(\rho C)^{\text{eff}*}$. Consequently, the solution of the above boundary value problem (C.35)-(C.38) appears as a linear function of the macroscopic gradient of temperature, modulo an arbitrary function $\widetilde{T}^{*(1)}(\mathbf{x}^*, t)$ [*Auriault et al.*, 2009; *Geindreau and Auriault*, 2001], and is written:

$$T_i^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{m}_i^*(\mathbf{y}^*, \mathbf{grad}_{x^*} p_a^{*(0)}) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_i^{*(1)} \tag{C.39}$$

$$T_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{m}_a^*(\mathbf{y}^*, \mathbf{grad}_{x^*} p_a^{*(0)}) \cdot \mathbf{grad}_{x^*} T^{*(0)} + \widetilde{T}_a^{*(1)} \tag{C.40}$$

where $\mathbf{m}_i^*(\mathbf{y}^*, \mathbf{grad}_{x^*} p_a^{*(0)})$ and $\mathbf{m}_a^*(\mathbf{y}^*, \mathbf{grad}_{x^*} p_a^{*(0)})$ are two periodic vectors which characterize the fluctuation of temperature in both phases at the pore scale. They also depend on the velocity field at the first order and hence on the macroscopic pressure gradient $\mathbf{grad}_{x^*} p_a^{*(0)}$. Introducing (C.39) and (C.40) in the set (C.35)-(C.38), these two vectors are solution of the following boundary value problem in a compact form:

$$-\beta \langle \mathbf{v}_a^{*(0)} \rangle - \text{div}_{y^*}(\kappa_i^*(\mathbf{grad}_{y^*} \mathbf{m}_i^* + \mathbf{I})) = \mathbf{0} \quad \text{in } \Omega_i \tag{C.41}$$

$$-\gamma \langle \mathbf{v}_a^{*(0)} \rangle + \rho_a^* C_a^* \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{y^*} \mathbf{m}_a^* + \mathbf{I}) - \text{div}_{y^*}(\kappa_a^*(\mathbf{grad}_{y^*} \mathbf{m}_a^* + \mathbf{I})) = \mathbf{0} \quad \text{in } \Omega_a \tag{C.42}$$

$$\mathbf{m}_i^* = \mathbf{m}_a^* \quad \text{on } \Gamma \tag{C.43}$$

$$(\kappa_i^*(\mathbf{grad}_{y^*} \mathbf{m}_i^* + \mathbf{I}) - \kappa_a^*(\mathbf{grad}_{y^*} \mathbf{m}_a^* + \mathbf{I})) \cdot \mathbf{n}^{\mathbf{\Gamma}} = \mathbf{0} \quad \text{on } \Gamma \tag{C.44}$$

$$\frac{1}{|\Omega|} \int_\Omega (\mathbf{m}_{\text{a}}^* + \mathbf{m}_{\text{i}}^*) \text{d}\Omega = \mathbf{0} \tag{C.45}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the equations (C.3, C.4, C.8, C.9) of order ε^2:

$$\rho_i^* C_i^* \frac{\partial T^{*(1)}}{\partial t^*} - \text{div}_{y^*}(\kappa_i^*(\mathbf{grad}_{y^*} T_i^{*(2)} + \mathbf{grad}_{x^*} T_i^{*(1)}))$$

$$-\mathrm{div}_{x^*}(\kappa_i^*(\mathbf{grad}_{y^*}T_i^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_i \tag{C.46}$$

$$\rho_a^* C_a^* \frac{\partial T^{*(1)}}{\partial t^*} + \rho_a^* C_a^* \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{y^*}T^{*(2)} + \mathbf{grad}_{x^*}T^{*(1)}) + \rho_a^* C_a^* \mathbf{v}_a^{*(1)} \cdot (\mathbf{grad}_{y^*}T^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})$$

$$-\mathrm{div}_{y^*}(\kappa_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)})) - \mathrm{div}_{x^*}(\kappa_a^*(\mathbf{grad}_{y^*}T_a^{*(1)} + \mathbf{grad}_{x^*}T^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C.47}$$

$$T_i^{*(2)} = T_a^{*(2)} \quad \text{on } \Gamma \tag{C.48}$$

$$(\kappa_i^*(\mathbf{grad}_{y^*}T_i^{*(2)} + \mathbf{grad}_{x^*}T_i^{*(1)}) - \kappa_a^*(\mathbf{grad}_{y^*}T_a^{*(2)} + \mathbf{grad}_{x^*}T_a^{*(1)})) \cdot \mathbf{n}^{\mathbf{\Gamma}} = L_{sg}^* w_n^{*(0)} \quad \text{on } \Gamma \tag{C.49}$$

where the unknowns $T_i^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ and $T_a^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ are $\mathbf{y}^*$-periodic. The fluid velocity $v_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is given by the equation (C.19) (and verifies the relation (C.16)) and $v_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ verifies the relations (C.21) and (C.22). Finally, $w_n^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is the normal interface velocity due to the phase change at the zero order given by the Hertz-Knudsen equation (B.67) and the Clausius Clapeyron's law (B.66). Integrating (C.46) over Ω_i and (C.47) over Ω_a, and then using the divergence theorem, the periodicity condition, and the boundary conditions (C.49) lead to the first order correction:

$$(\rho C)^{\mathrm{eff}*} \frac{\partial T^{*(1)}}{\partial t^*} + \rho_a^* C_a^* \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} \widetilde{T}_i^{*(1)} + \rho_a^* C_a^* \langle \mathbf{v}_a^{*(1)} \rangle \cdot \mathbf{grad}_{x^*} T_i^{*(0)}$$

$$-\mathrm{div}_{x^*}(\mathbf{k}^{\mathrm{disp}*} \mathbf{grad}_{x^*}\ T^{*(0)}) = \mathrm{SSA}_V L_{sg}^* w_n^{*(0)} \tag{C.50}$$

where $\mathbf{k}^{\mathrm{dips}*}$ is the effective thermal dispersion tensor respectively, defined as:

$$\mathbf{k}^{\mathrm{disp}*} = \frac{1}{|\Omega|} \left(\int_{\Omega_\mathrm{a}} \kappa_a^*(\mathbf{grad}_{y^*}\mathbf{m}_a^*(\mathbf{y}^*) + \mathbf{I}) + \mathbf{v}_a^{*(0)} \otimes \mathbf{m}_a^* \mathrm{d}\Omega + \int_{\Omega_\mathrm{i}} \kappa_i^*(\mathbf{grad}_{y^*}\mathbf{m}_i^*(\mathbf{y}^*) + \mathbf{I}) \mathrm{d}\ \Omega \right) \tag{C.51}$$

Finally, we can define:

$$\langle T^* \rangle = T^{*(0)} + \varepsilon \widetilde{T}^{*(1)}, \quad \langle \mathbf{v}_a^* \rangle = \langle \mathbf{v}_a^{*(0)} \rangle + \varepsilon \langle \mathbf{v}_a^{*(1)} \rangle \tag{C.52}$$

where $\langle\cdot\rangle$ represents the mean over the REV. Thus, adding equations (C.34) and (C.50) multiplied by ε, we get the following dimensionless macroscopic description at the second order of approximation:

$$(\rho C)^{\text{eff}*}\frac{\partial\langle T^*\rangle}{\partial t^*}+\rho_a^* C_a^*\langle\mathbf{v}_a^*\rangle\cdot\mathbf{grad}_{x^*}\langle T^*\rangle-\text{div}_{x^*}(\varepsilon\mathbf{k}^{\text{disp}*}\mathbf{grad}_{x^*}\langle T^*\rangle)=\text{SSA}_V L_{sg}^* w_n^{*(0)} \quad \text{(C.53)}$$

Water vapor transfer

Introducing asymptotic expansions for ρ_v^* in the relations (C.5, C.10) give at the lowest order (ε^0)

$$\mathbf{v}_a^{*(0)}\cdot\mathbf{grad}_{y^*}\rho_v^{*(0)}-\text{div}_{y^*}(D_v^*\mathbf{grad}_{y^*}\rho_v^{*(0)})=0 \quad \text{in } \Omega_a \quad \text{(C.54)}$$

$$D_v^*\mathbf{grad}_{y^*}\rho_v^{*(0)}\cdot\mathbf{n}^{\boldsymbol{\Gamma}}=0 \quad \text{on } \Gamma. \quad \text{(C.55)}$$

where the unknown $\rho_v^{*(0)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is $\mathbf{y}^*$-periodic. It can be shown [*Auriault et al.*, 2009] that the solution of the above boundary value problem is given by:

$$\rho_v^{*(0)}=\rho_v^{*(0)}(\mathbf{x}^*,t). \quad \text{(C.56)}$$

At the first order, the water vapor density is independent of the microscopic dimensionless variable $\mathbf{y}^*$. Taking into account these results, the second-order problem is given by the equations (C.5, C.10) of order ε:

$$\frac{\partial\rho_v^{*(0)}}{\partial t^*}+\mathbf{v}_a^{*(0)}\cdot(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)})-\text{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)}))=0 \quad \text{in } \Omega_a \quad \text{(C.57)}$$

$$D_v^*(\mathbf{grad}_{y^*}\rho_v^{*(1)}+\mathbf{grad}_{x^*}\rho_v^{*(0)})\cdot\mathbf{n}^{\boldsymbol{\Gamma}}=0 \quad \text{on } \Gamma. \quad \text{(C.58)}$$

where the unknown $\rho_v^{*(1)}(\mathbf{x}^*,\mathbf{y}^*,t)$ is $\mathbf{y}^*$-periodic and the macroscopic gradient $\mathbf{grad}_{x^*}\rho_v^{*(0)}$ is given. $\mathbf{v}_a^{*(0)}$ is given by the relation (C.19) (and verifies the relations (C.16) and (C.18)). Integrating (C.57) over Ω_a and taking into account the condition of periodicity, the boundary condition (C.58) and the relation (C.19), we obtain the following first order macroscopic description:

$$\phi \frac{\partial \rho_v^{*(0)}}{\partial t^*} + \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)} = 0 \tag{C.59}$$

where $\langle \mathbf{v}_a^{*(0)} \rangle$ is given by the relation (C.24).

As for the temperature, the convection alone is present at the first order of approximation. Using the relation (C.59), the boundary value problem (C.57)-(C.58) is written:

$$-\phi^{-1} \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)} + \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})$$

$$-\mathrm{div}_{y^*}(D_v^* (\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C.60}$$

$$D_v^* (\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)}) \cdot \mathbf{n}^{\boldsymbol{\Gamma}} = 0 \quad \text{on } \Gamma. \tag{C.61}$$

The solution of the above boundary value problem appears as a linear function of the macroscopic gradient of the water vapor, modulo an arbitrary function $\widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t)$ [*Auriault et al.*, 2009]:

$$\rho_v^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t) = \mathbf{h}_v^*(\mathbf{y}^*, \mathbf{grad}_{x^*} p_a^{*(0)}) \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)} + \widetilde{\rho}_v^{*(1)}(\mathbf{x}^*, t) \tag{C.62}$$

where $\mathbf{h}_v^*(\mathbf{y}^*, \mathbf{grad}_{x^*} p_a^{*(0)})$ is a periodic vector which characterizes the fluctuation of water vapor density in the air phase at the pore scale which depends on the intensity of the flow and thus on $\mathbf{grad}_{x^*} p_a^{*(0)}$. Introducing (C.62) in the set (C.60)-(C.61), this vector is solution of the following boundary value problem in a compact form:

$$-\phi^{-1} \langle \mathbf{v}_a^{*(0)} \rangle + \mathbf{v}_a^{*(0)} + \mathbf{v}_a^{*(0)} \cdot \mathbf{grad}_{y^*} \mathbf{h}_v^* - \mathrm{div}_{y^*}(D_v^* (\mathbf{grad}_{y^*} \mathbf{h}_v^* + \mathbf{I})) = \mathbf{0} \quad \text{in } \Omega_a \tag{C.63}$$

$$D_v^* (\mathbf{grad}_{y^*} \mathbf{h}_v^* + \mathbf{I}) \cdot \mathbf{n}^{\boldsymbol{\Gamma}} = \mathbf{0} \quad \text{on } \Gamma \tag{C.64}$$

$$\frac{1}{|\Omega|} \int_{\Omega_a} \mathbf{h}_v^* \mathrm{d}\Omega = \mathbf{0} \tag{C.65}$$

This latter equation is introduced to ensure the uniqueness of the solution. Finally, the third order problem is given by the equations (C.5, C.10) of order ε^2:

$$\frac{\partial \rho_v^{*(1)}}{\partial t^*} + \mathbf{v}_a^{*(0)} \cdot (\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) + \mathbf{v}_a^{*(1)} \cdot (\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})$$

$$-\mathrm{div}_{y^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)})) - \mathrm{div}_{x^*}(D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(1)} + \mathbf{grad}_{x^*} \rho_v^{*(0)})) = 0 \quad \text{in } \Omega_a \tag{C.66}$$

$$D_v^*(\mathbf{grad}_{y^*} \rho_v^{*(2)} + \mathbf{grad}_{x^*} \rho_v^{*(1)}) \cdot \mathbf{n}^{\mathbf{\Gamma}} = \rho_i^* w_n^{*(0)} \quad \text{on } \Gamma \tag{C.67}$$

where the unknown $\rho_v^{*(2)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is $\mathbf{y}^*$-periodic. The fluid velocity $v_a^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is given by the equation (C.19) (and verifies the relation (C.16)) and $v_a^{*(1)}(\mathbf{x}^*, \mathbf{y}^*, t)$ verifies the relations (C.21) and (C.22). Finally, $w_n^{*(0)}(\mathbf{x}^*, \mathbf{y}^*, t)$ is the normal interface velocity due to the phase change at the zero order given by the Hertz-Knudsen equation (B.67) and the Clausius Clapeyron's law (B.66). Integrating (C.66) over Ω_a, and then using the divergence theorem, the periodicity condition, and the boundary conditions (C.67) lead to the first order correction:

$$\phi \frac{\partial \rho_v^{*(1)}}{\partial t} + \langle \mathbf{v}_a^{*(0)} \rangle \cdot \mathbf{grad}_{x^*} \widetilde{\rho}_v^{*(1)} + \langle \mathbf{v}_a^{*(1)} \rangle \cdot \mathbf{grad}_{x^*} \rho_v^{*(0)}$$

$$-\mathrm{div}_{x^*}(\mathbf{D}^{\mathrm{disp}*} \mathbf{grad}_{x^*} \rho_v^{*(0)}) = -\mathrm{SSA}_V \rho_i^* w_n^{*(0)} \tag{C.68}$$

where $\mathrm{SSA_V} = |\Gamma|/|\Omega|$ is the surface area and $\mathbf{D}^{\mathrm{disp}*}$ is the dimensionless effective dispersion tensor for the water vapor defined as:

$$\mathbf{D}^{\mathrm{disp}*} = \frac{1}{|\Omega|} \int_{\Omega_a} D_v^*(\mathbf{grad}_{y^*}\ \mathbf{h}_v^*(\mathbf{y}^*) + \mathbf{I}) + \mathbf{v}_a^{*(0)} \otimes \mathbf{h}_v^* \mathrm{d}\Omega \tag{C.69}$$

Finally, we can define:

$$\langle \rho_v^* \rangle_a = \rho_v^{*(0)} + \varepsilon \widetilde{\rho}_v^{*(1)}, \quad \langle \mathbf{v}_a^* \rangle = \langle \mathbf{v}_a^{*(0)} \rangle + \varepsilon \langle \mathbf{v}_a^{*(1)} \rangle \tag{C.70}$$

where $\langle\cdot\rangle_a$ and $\langle\cdot\rangle$ represent the mean over the air phase and over the REV respectively. Thus, adding equations (C.59) and (C.68) multiplied by ε, we get the following dimensionless macroscopic description at the second order of approximation:

$$\phi\frac{\partial\langle\rho_v^*\rangle_a}{\partial t} + \langle\mathbf{v}_a^*\rangle\cdot\mathbf{grad}_{x^*}\langle\rho_v^*\rangle_a - \mathrm{div}_{x^*}(\varepsilon\mathbf{D}^{\mathrm{disp}*}\mathbf{grad}_{x^*}\langle\rho_v^*\rangle_a) = -\mathrm{SSA}_V\rho_i^* w_n^{*(0)} \qquad \text{(C.71)}$$

References

Auriault, J.-L., C. Boutin, and C. Geindreau. (2009), *Homogenization of Coupled Phenomena in Heterogenous Media*, Wiley-ISTE, London.

Calonne, N., C. Geindreau, and F. Flin (2014), Macroscopic modeling for heat and water vapor transfer in dry snow by homogenization, *The Journal of Physical Chemistry B*, *118*(47), 13,393–13,403, doi:10.1021/jp5052535.

Geindreau, C., and J.-L. Auriault (2001), Transport phenomena in saturated porous media under liquid-solid phase change, *Arch. Mech.*, *53*, 385–420.