Numerical filters: Generalities

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Contents

Why do we need filters?

In several places in NWP one is confronted with the problem of removing parts of the spectrum that are clearly unwanted:

- UNSTABLE physical gravity waves
- computational SPURIOUS modes

These parts may be unwanted because:

- we want to save computation COST, , e.g. SI scheme unstable modes allows us to use a bigger time step.
- they may SPOIL the forecast: e.g. spurious reflection caused by using ill-posed boundary conditions.
- ESTHETICS-kind of reason: we don't like noisy features on the maps.

Definitions

Kulhanek:

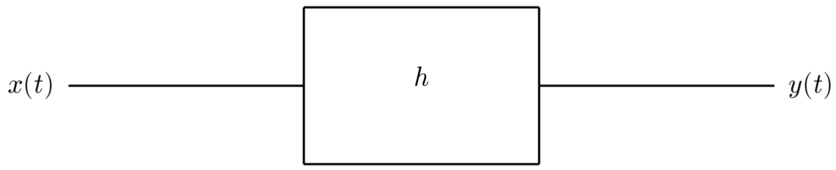
> *The term* `filter` *is used to characterize a system which can perform an effective prescribed separation of the desired information carried by the signal from the unwanted portion commonly called* `noise`.

We shall use the term noise for any part of the signal that is unwanted.

A filter will transform an signal $x(t)$ into an output signal $y(t)$ by applying an *operator* h to it:

$$y(t) = h\left[x(t)\right] \ . \tag{1}$$

Setting the stage: notions



Let us, for the sake of the argument, assume:

1. The operator is LINEAR:
   $h[x_1 + x_2] = h[x_1] + h[x_2]$.

2. The operator is TRANSLATION INVARIANT:
   $(\partial h / \partial t)\,[x(t)] = 0$.

CALL $h(t) \equiv h[\delta(t)]$.

TERMINOLOGY $h(t)$ = **response** function.

translation invariance $\Rightarrow h[\delta(t-\tau)] = h(t-\tau)$.

A signal is decomposed in $x(\tau)\delta(t-\tau)$:

$$\textit{linearity} \Rightarrow y(t) = \int_{-\infty}^{\infty} d\tau x(\tau) h(t-\tau) \qquad (2)$$

$\equiv$ **convolution**

Setting the stage: notions

- causal:

$$h(t) = 0 \quad \text{if} \quad t < 0\,.$$

- stable: FINITE INPUT $\Rightarrow$ FINITE OUTPUT:

$$\begin{aligned} |y(t)| &= \left|\int_{-\infty}^{\infty} d\tau\, x(\tau)\, h(t-\tau)\right| \\ &\leq \int_{-\infty}^{\infty} d\tau\, |x(\tau)|\, |h(t-\tau)| \\ &\leq \max\left(|x(t)|\right) \int_{-\infty}^{\infty} dt\, |h(t)|\,. \end{aligned} \tag{3}$$

$$\Leftrightarrow \int_{-\infty}^{\infty} dt\, |h(t)| < \infty\,. \tag{4}$$

Set the stage: spectral form

Conventions:

$$X(\omega) \equiv \int dt\, x(t)\, e^{-i\omega t} \qquad x(t) = \frac{1}{2\pi}\int d\omega\, X(\omega)\, e^{i\omega t}$$

Plug in $x(\tau)$ and $h(t-\tau)$ in the convolution and use $\frac{1}{2\pi}\int d\omega\, e^{i\omega t} = \delta(t)$ then get

$$Y(\omega) = H(\omega)\, X(\omega)\,. \tag{5}$$

In spectral form, the CONVOLUTION *becomes a* PRODUCT

$$x(t) = A\,\sin(\Omega t)\,, \quad H(\omega) = M(\omega)\, e^{i\phi(\omega)}\,. \tag{6}$$

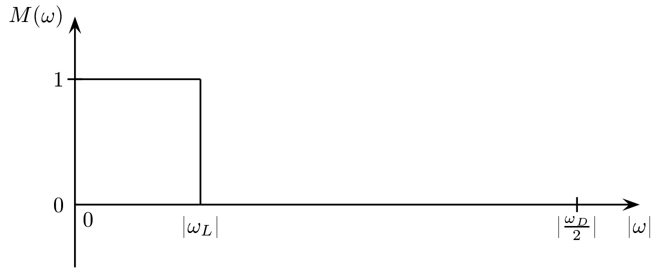
then

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int d\omega\, X(\omega)\, H(\omega)\, e^{i\omega t}\,, \\ &= A\, M(\Omega)\, \sin\left(\Omega t + \phi(\Omega)\right)\,. \end{aligned} \tag{7}$$

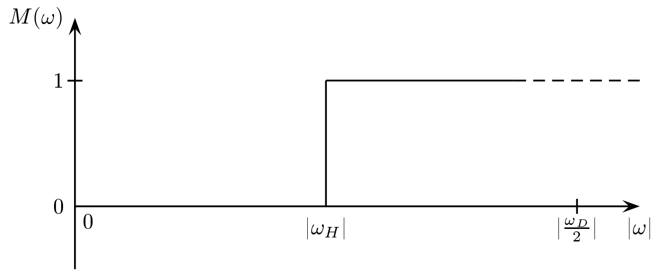
The reponse to a wave is again a wave

Set the stage: types

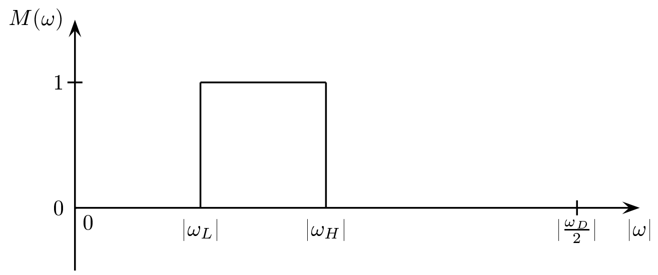
- Low-pass filter



- High-pass filter



- Band-pass filter



NWP: discretization

In NWP we only deal with discretized fields,

$$x(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\,\delta(t - n\Delta t)$$

Calling $x_n \equiv x(n\Delta t)$ and Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x_n\, e^{-i\omega n\Delta t}$$

$$(\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots)$$

$$\ldots \quad -2\Delta t, \quad -\Delta t, \quad 0, \quad \Delta t, \quad 2\Delta t, \quad 3\Delta t, \quad \ldots$$

$$\boxed{\text{PERIODIC: } X(\omega) = X(\omega + \omega_D) \text{ with } \omega_D = \frac{2\pi}{\Delta t}}$$

$$x_n = \frac{1}{\omega_D} \int_{-\frac{\omega_D}{2}}^{\frac{\omega_D}{2}} X(\omega)\, e^{i\omega n\Delta t}$$

Idem

$$h(t) \to (\ldots, h_{-1}, h_0, h_1, h_2, h_3, \ldots)$$
$$y(t) \to (\ldots, y_{-1}, y_0, y_1, y_2, y_3, \ldots)$$

Discretized convolution:

$$y_n = \sum_{k=-\infty}^{\infty} x_k \, h_{n-k}$$

The following notation is often used: $z = e^{i\omega\Delta t}$. Then

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \, z^{-n} \, .$$

In this notation the convolution becomes:

$$Y(z) = H(z)X(z) \, .$$

Discretize analog ↔ digital

Analog	Digital
$X(\omega) \equiv \int dt\, x(t)\, e^{-i\omega t}$	$X(\omega) = \sum x_n\, e^{-i\omega n \Delta t}$
$X(-\infty \leftarrow \omega \rightarrow \infty)$	$\begin{cases} X(\omega + \omega_D) = X(\omega) \\ \omega_D = \frac{2\pi}{\Delta t} \end{cases}$
$h(t) \equiv h[\delta(t)]$	$h_n \equiv h[(1, 0, 0, \ldots)]$
$y(t) = \int d\tau x(\tau) h(t - \tau)$	$y_n = \sum_k x_k h_{n-k}$
$\int_{-\infty}^{\infty} dt\, \lvert h(t) \rvert < \infty$	$\sum_n \lvert h_n \rvert < \infty$

Truncation

Take the example of a *real* response function.

Then

$$h_{-n} = h_n$$

and the response function becomes

$$H(\omega) = h_0 + 2\sum_{n=1}^{\infty} h_n \cos \omega n \Delta t\,.$$

If stable

$$|h_0| + 2\sum_{n=1}^{\infty} |h_n| < \infty \Rightarrow h_n \to_{n\to\infty} 0$$

Then we can *truncate* the expansion.

$$H(\omega) = h_0 + 2\sum_{n=1}^{N} h_n \cos \omega n \Delta t\,.$$

This is NECESSARY for an implementation on a COMPUTER!

The effect of truncation

Let us consider a LOW-PASS filter

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_L \\ 0 & \text{for } |\omega| > \omega_L \end{cases}$$

The coefficients become:

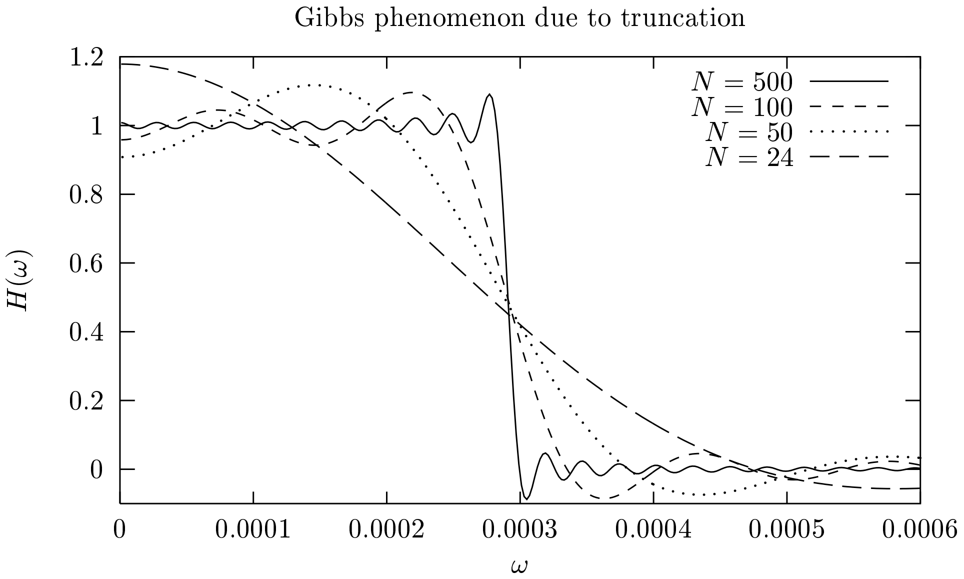
$$h_n = \frac{\sin n\omega_L \Delta t}{n\pi}$$

with $\Delta t = 450s$, $\omega_L = \frac{2\pi}{6h}$ and $N = 24, 50, 100, 500$

conclusion:

The effect of the TRUNCATION *of the infinite expansion to a* FINITE *expansion that can be handled by a* COMPUTER *is the generation of* GIBBS WAVES *at the discontinuity*

Gibbs phenomenon



Removing Gibbs waves by a spectral window

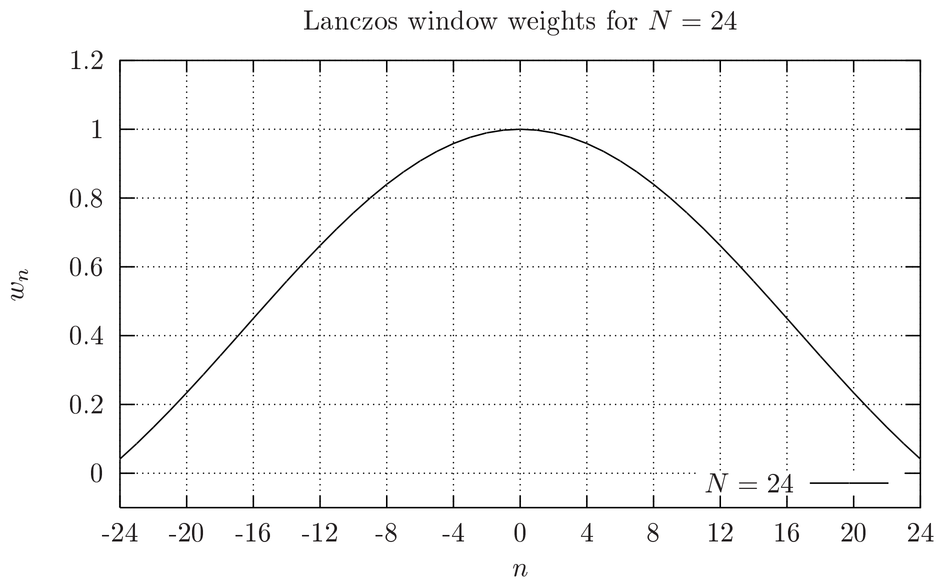
We introduce weights w_n and mulitply the components by them

$$(\dots, w_{-1}h_{-1}, w_0 h_0, w_1 h_1, w_2 h_2, w_3 h_3, \dots) \tag{8}$$

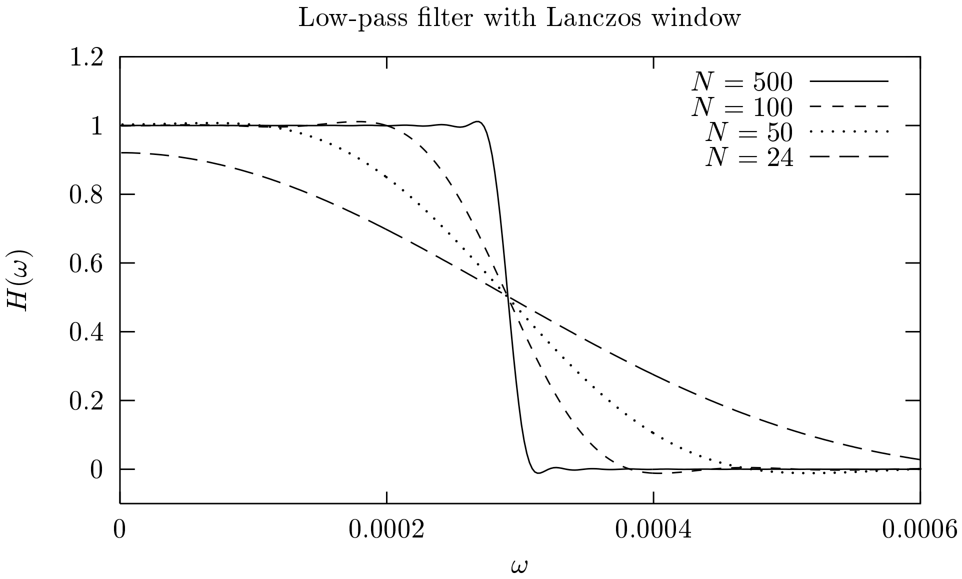
$$H_L(\omega) = w_0 h_0 + 2\sum_{n=1}^{N} w_n h_n \cos \omega n \Delta t \,. \tag{9}$$

example: Lanczos window:

$$w_n = \frac{\sin \frac{n\pi}{N+1}}{\frac{n\pi}{N+1}} \tag{10}$$



Ideal low-pass with Lanczos window



Discussion discretization and truncation

Discretization:

- spectral representation becomes *periodic*: this corresponds to a lowest scale that can be discribed (*resolution*)

Truncation:

- *possible* if filter is *stable*
- causes *Gibbs waves* at the *discontinuities*

Gibbs wave can be treated by

- introducing a *spectral window*
- taking *longer cut off* of the truncation, see next topic: *recursive filters*.

Increased efficiency by recursive filters

RECURSIVE FILTERS are:

$$y_n = \sum_{k=0}^{N} a_k \, x_{n-k} - \sum_{l=1}^{L} b_l \, y_{n-l}$$

L is called the *order* of the filter. For example,

$$y_n = a \, x_n - b \, y_{n-1} \, , \quad y_{-n} = 0 \quad \text{for } n \geq 0$$

$$\begin{aligned} y_0 &= ax_0 \, , \\ y_1 &= ax_1 - by_0 = ax_1 - abx_0 \, , \\ y_2 &= ax_2 - by_1 = ax_2 - abx_1 + ab^2 x_0 \, , \\ &\vdots \\ y_n &= ax_n - abx_{n-1} + \ldots + (-1)^n \, ab^n x_0 \, . \end{aligned}$$

Response for $a = 1$, $b = 0.9$:

$$\begin{aligned} h &= h[(1, 0, 0, \ldots)] = (a, -ab, ab^2, -ab^3, \ldots) \\ &= (1, -0.900, 0.810, -0.729, 0.656, \ldots) \end{aligned}$$

$h < 0.1$: 21 operations $\leftrightarrow$ 2 recursively

MORE EFFICIENT COMPUTING!

Example: Butterworth

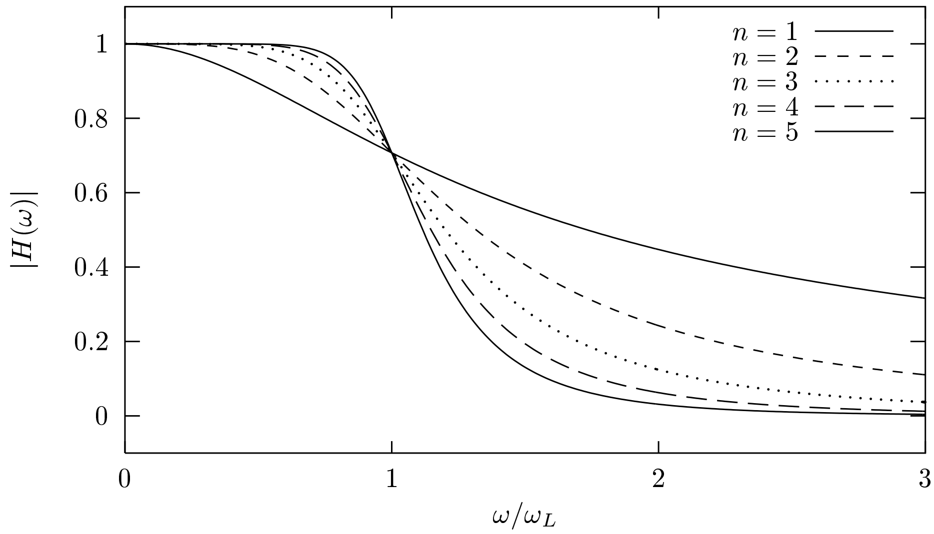
Defined by the *square* of the response

$$|H(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_L}\right)^{2n}}$$

where n is called the order of the filter.

Let us normalize $\nu = \frac{\omega}{\omega_L}$

Response of Butterworth filters of different order

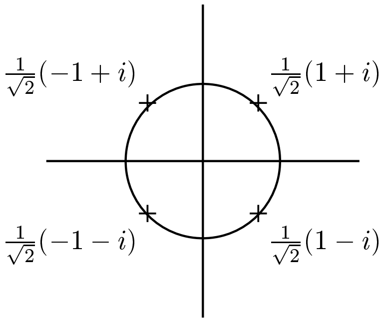


The filter is becomes a BETTER APPROXIMATION of the ideal low-pass as $n \to \infty$

2^{nd}-order Butterworth

1. search the poles of the denominator of $\frac{1}{1+\nu^4}$:

$$\begin{aligned}\nu_{1,2} &= \frac{1}{\sqrt{2}}(1 \pm i)\\ \nu_{3,4} &= \frac{1}{\sqrt{2}}(-1 \pm i)\end{aligned}$$

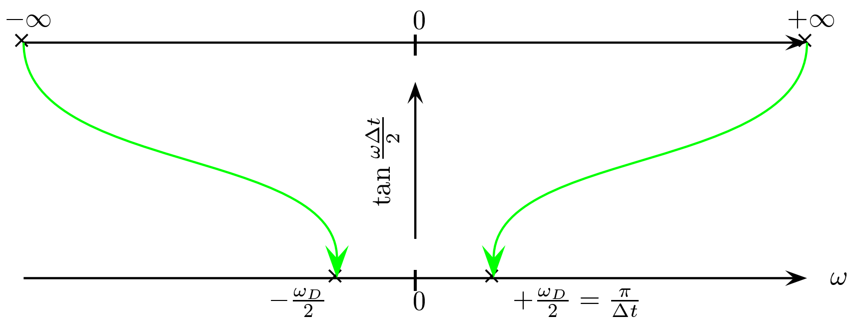


To take the *square root* it is sufficient to take *two poles* in the upper half of the complex plane:

$$\begin{aligned}H(\nu) &= \frac{-1}{\left(\nu - \frac{1}{\sqrt{2}}(1+i)\right)\left(\nu - \frac{1}{\sqrt{2}}(-1+i)\right)}\\ &= \frac{1}{(i\nu)^2 + \sqrt{2}\,(i\nu) + 1}\end{aligned}$$

Design a 2^{nd}-order digital Butterworth

RECALL: that in the discretized case, the spectral form is *periodic*. Can be done by a mapping



Notice that, using $z = e^{i\omega \Delta t}$,

$$i \tan\left(\frac{\omega \Delta t}{2}\right) = \frac{1 - z^{-1}}{1 + z^{-1}}, \quad \omega_A = \tan \frac{\omega_L \Delta t}{2}.$$

The mapping is done by replacing

$$i\,\nu = i\frac{\omega}{\omega_L} \to \frac{1 - z^{-1}}{1 + z^{-1}} \frac{1}{\omega_A}$$

$$H(z) = \frac{\left(\frac{\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}\right)\left(1 + 2\,z^{-1} + z^{-2}\right)}{1 + \frac{2\left(\omega_A^2-1\right)}{1+\sqrt{2}\omega_A+\omega_A^2}\,z^{-1} + \frac{1-\sqrt{2}\omega_A+\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}\,z^{-2}}$$

$2^{nd} - order filter$

Has the form

$$H(z) = \frac{\sum_{k=0}^{N} a_k \, z^{-k}}{1 + \sum_{l=1}^{L} b_l \, z^{-l}}$$

Using

$$Y(z) = H(z)X(z)\,.$$

this can be arranged as

$$y_n = \sum_{k=0}^{2} a_k \, x_{n-k} - \sum_{l=1}^{2} b_l \, y_{n-l}$$

and is indeed second-order!

coefficients

$$
\begin{aligned}
a_0 &= \frac{\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}\,, \\
a_1 &= 2\,\frac{\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}\,, \\
a_2 &= \frac{\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}\,, \\
b_1 &= \frac{2\left(\omega_A^2-1\right)}{1+\sqrt{2}\omega_A+\omega_A^2}\,, \\
b_2 &= \frac{1-\sqrt{2}\omega_A+\omega_A^2}{1+\sqrt{2}\omega_A+\omega_A^2}\,,
\end{aligned}
$$

Response function

Then response function can be calculated for $\omega_L = 1$ and for different resolutions Δt:

- $\Delta t = 1.571 = \pi/2 \Rightarrow \omega_D = 4$

  $$h_n = (0.2929, 0.5858, 0.2426, -0.1005, \ldots)$$

- $\Delta t = 1.047 = \pi/3 \Rightarrow \omega_D = 6$

  $$h_n = (0.1551, 0.4063, 0.3697, 0.1316, \ldots)$$

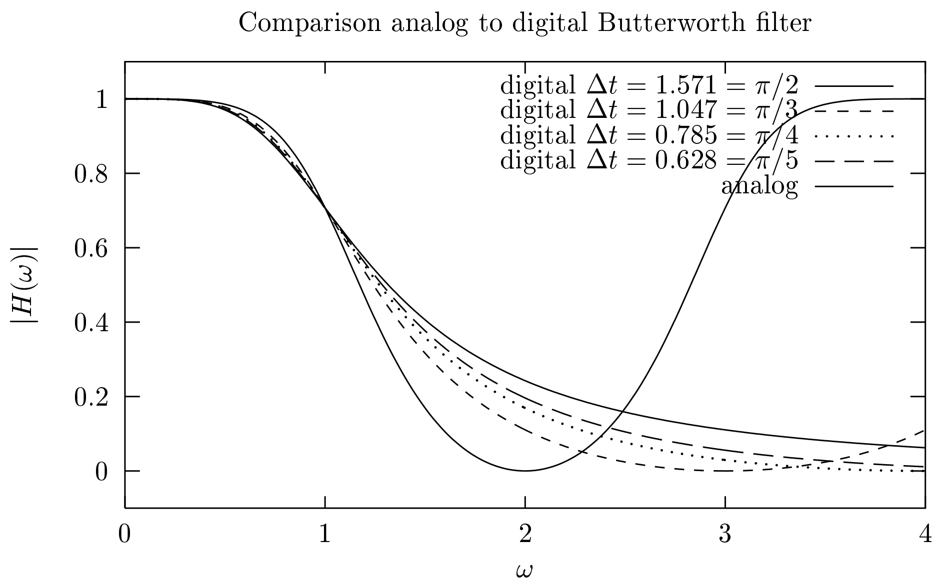
- $\Delta t = 0.785 = \pi/4 \Rightarrow \omega_D = 8$

  $$h_n = (0.0976, 0.2873, 0.3360, 0.2210, \ldots)$$

- $\Delta t = 0.628 = \pi/5 \Rightarrow \omega_D = 10$

  $$h_n = (0.0675, 0.2120, 0.2819, 0.2347, \ldots)$$

Comparison digital–analog Butterworth



We have:

- shown the subleties when implementing a filter NUMERICALLY
- shown the effects of DISCRETIZING AND TRUNCATING: Gibbs
- explained how this can be circumvented by
  1. increasing computing cost
  2. RECURSIVE filters
- given an example of how a digital filter can be DESIGNED