Design of the hybrid vertical coordinate η (case of a domain with $\pi_{\text{top}} \neq 0$)

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1 Introduction

Officially, in the η hybrid coordinate, the functions A and B are said "arbitrary" for the definition of an hybrid coordinate. However, even if A and B are chosen such as :

$$A(0) = 0 \quad (1)$$
$$A(1) = 0 \quad (2)$$
$$B(0) = 0 \quad (3)$$
$$B(1) = 1 \quad (4)$$

and even if B is taken positive and monotonously growing and A positive, this does not guarantee that this choice results in a coordinate (i.e. that $\pi(\eta) = A(\eta) + B(\eta)\pi_s$ is monotonous in η). It should be noticed here that as soon A is not identically zero, there always exists a surface-pressure value below which a given choice of (A, B) ceases to be a coordinate. This value is given by :

$$\pi_s = \max_{\eta \in [0,1]} \left[\frac{-dA/d\eta}{dB/d\eta} \right] \quad (5)$$

The coordinate property is then violated in the area where $dA/d\eta$ is negative and large, i.e. in the area where the "terrain-followingness" exhibits its fastest decrease. However, for applications to meteorology, we know that π_s cannot be smaller than an absolute given limite $\pi_s(\text{Everest})$, hence we are allowed to choose coordinates for which A is not identically zero, that is, coordinates which actually are hybrid coordinates, provided that (5) cannot be satisfied.
Currently, in the ARPEGE and Aladin models, this constraint is set by aborting if A and B are such that the following condition is violated :

$$\max \left[\frac{-dA/d\eta}{dB/d\eta} \right] > \pi_{\text{sMin}} \qquad [\text{condition for arborting}] \quad (6)$$

where π_{sMin} is currently set to 450 hPa. This condition is currently checked through a test in FACADI.F90, that is, very late, in the model itself.

Note : Since this value 450 hPa is larger than the effective $\pi_s(\text{Everest})$, this constraint should be made more stringent in the future, when the resolution will increase so as the effective height of the Mount Everest will pass below 450 hPa.

Currently, the specification of A and B is made by using an external program, in which some basic parameters are specified as an input. However, the current programm does not allow all the flexibility that could be desired for such a tool, and more annoying, it does not prevent outputs of A, B which are not coordinates for the model in practice. The constraint (6) is ignored by this program, and even if it was diagnosed, the program does not indicate how to modify a bad choice of A, B in order to obtain a valid coordinate.
The aim of this paper is to present an alternative way to define the functions A and B, allowing more flexibility than with the current method, and allowing an intuitive way to correct an ill-defined initial coordinate.

N.B. : In addition to provide a more flexible tool, the present paper is designed in order to allow "vertically bounded" domains, that is A and/or B are allowed to be non zero at top. A more simple version of this documentation (file memoeta0.tex) exists for the case of "vertically unbounded" domains only, that is, for which the top pressure is identically zero.

2 Guideline

Three basic parameters are first set : the number of half level for the output $L+1$ (where L is the number of layers of full levels), a reference pressure $\pi_{00} = 101325.$ Pa, and the minimum surface pressure π_{sMin} for which the set A, B must define a valid coordinate.
In order to practically define the namelists, an level index $\tilde{l}$ is introduced. By definition we have :

$$\tilde{l} \in 0, 1, ..., L. \tag{7}$$

In the ARPGE/IFS terminology, the index $\tilde{l}$ describes the index of half levels. The index $\tilde{l} = 0$ represents the top of the domain, while $\tilde{l} = L$ represents the bottom of the domain.

However, the A and B functions will be defined through continuous functions. Therefore a new independant variable has to be defined. A real independant variable x is introduced, with

$$x \in [0, 1]. \tag{8}$$

The value $x = 0$ represents the top of the domain, while $x = 1$ represents the bottom of the domain. Once continuous versions of A and B will have been defined, we will finally discretise them through :

$$x(\tilde{l}) = \tilde{l}/L \tag{9}$$

The constraint (6) is a given fact that we have to take into account as soon as in the definition of A, B functions. The principle of the present approach to fullfil this requirement is to define :

- A "stretching" function : this function defines how the resolution of the η coordinate should vary along the vertical when $\pi_s = \pi_{00}$. Let $m(x)$ be this function, referred to as the "stretching function" hereafter. This stretching function is thus defined by :

$$m : [0,1] \longrightarrow [y_T, 1] \tag{10}$$
$$x \longrightarrow y = m(x) \tag{11}$$

  where :

$$0 \le y_T < 1 \tag{12}$$

  The free parameter y_T is used to specify portion of the atmosphere which must be excluded from the domain when $\pi_s = pi_{00}$. For instance, if $y_T = 0.1$ we will have $\pi_T = 0.1\pi_{00}$
- An "hybridicity" function : this function defines the "proportion" of terrain-following (σ) and pure-pressure (π) coordinates that we want to have at a given level y in the stretched grid. Let $h(y)$ be thus function, referred to as the "hybridicity function" hereafter. This hybridicity function is thus defined by :

$$h : [y_T, 1] \longrightarrow [z_T, 1] \tag{13}$$
$$y \longrightarrow z = h(y) \tag{14}$$

  where :

$$0 \le z_T \le y_T \tag{15}$$

  The z_T parameter is used to specify the hybridicity that is desired at the top of the domain. If $z_T = 0$, the top is a pure level pressure level, while if $z_T = y_T$, the top is a pure terrain-following level. In order to fulfil (15), the free parameter is overwritten to :

$$z_T = \min[y_T, \max(z_T, 0)] \tag{16}$$

Finally the A and B functions are defined directly from the stretching and the hybridicity functions :

$$A(x) = \pi_{00}\{m(x) - h[m(x)]\} \tag{17}$$
$$B(x) = h[m(x)] \tag{18}$$

For a given surface pressure π_s, the derivative of the pressure with respect to the coordinate writes, for a given level x_0 :

$$\frac{1}{\pi_{00}}\left(\frac{d\pi}{dx}\right)_{(x=x_0)} = \left(\frac{dm}{dx}\right)_{(x=x_0)}\left\{1 - \frac{(\pi_{00}-\pi_s)}{\pi_{00}}\left(\frac{dh}{dy}\right)_{(y=m[x_0])}\right\} \tag{19}$$

For this choice to define a coordinate under the constraint (6), it is therefore sufficient that h is chosen in such a way that :

$$\frac{dh}{dy} < X \equiv \frac{\pi_{00}}{(\pi_{00} - \pi_{\mathrm{sMin}})} \tag{20}$$

This defines a second constraint, that is easy to handle practically. The current value of X is 1.8 (from the test in `FACADI`).

The slope of the m function decribes the local vertical resolution in pressure. The value of m at the surface is chosen equal to $m(1) = 1$. At the top, for operational namelists, we currently choose to impose $m(0) = 0$, thus implicitly constraining the pressure to be zero at the top of the domain.
The "actual" hybridicity itselfwould be defined by $B/(B+A/\pi_{00})$ and thus given by $h(y)/y$, i.e. the slope of the straight-line joining the origin to the point $(y, h(y))$ on the curve of h (hence, $h(y)$ is *not* exactly the hybridicity itself but the hybridicity multiplied by y)· Since A and B are positive, we must have $h(y) \leq y$ for any value of y (this is why z_t has to be smaller than y_T). The proposed definition of $h(y)$ makes the expression of the constraint (20) more simple than what it would be with the exact hybridicity $h(y)/y$. In order to fullfil $\pi = \pi_s$ at the surface, we must impose $h(1) = 1$.

Except the constraints (20) and (12)–(15), the choice of m and h functions is now free. In order to restrict this hudge amount of degrees of freedom, we choose an approach in which some caracteristic points for these functions are explicitely specified. Then the functions are defined in such a way that they pass exactly by theses points with suitable continuity and derivability properties. For these specifications, the atmosphere is supposed to be discretized with L layers.

3 Definition of the stretching function

For the definition of the stretching function m, the vertical domain (consisting in L layer) is divided into 5 parts :
- the bottom model layer
- the "PBL"
- the "troposphere" (which is indeed the central part of the domain)
- the "stratosphere"
- the uppermost model layer

The four interfaces between these subdomains are then used as characteristic points, each of them being specified in practice through a level index l, and a corresponding pressure π (for a surface pressure $\pi_s = \pi_{00}$). Thus we define the following characteristic points :
- bottom of uppermost model layer : $x_1 = 1/L$, $y_1 = \delta pi_1/\pi_{00}$
- bottom of the "stratosphere" : $x_2 = (N_{\mathrm{Strato}})/L$, $y_2 = (\pi_{\mathrm{Strato}})/\pi_{00}$
- top of the "PBL" : $x_3 = (L - N_{\mathrm{PBL}})/L$, $y_3 = (\pi_{00} - \pi_{\mathrm{PBL}})/\pi_{00}$
- top of bottom model layer : $x_4 = (L-1)/L$, $y_4 = (\pi_{00} - \delta\pi_L)/\pi_{00}$

N.B. : Please note that if the specification of the uppermost characteristic point is to be specified from a desired value of π_1 at the first full level, then the knowledge of the value of `LAPRXPK` is required in order to know the corresponding pressure-depth of the first layer $\delta\pi_1$.

The stretching function is then defined sub-domain by sub-domain :

uppermost sub-domain (for $0 < x \leq x_1$) :

$m(x) = y_T + (y1/x1)\,x$ (used only if an analytic expression of A, B is needed)

"Strato" sub-domain : (for $x_1 < x \leq x_2$) :

We note $d_1 = \dfrac{x_1 y_2 - x_2 y_1 + y_T(x_2 - x_1)}{x_1}\,(x_2 - x_1)^{-\alpha_1}$

$m(x) = y_T + x\,(y1 - y_T)/x1 + d_1\,(x - x_1)^{\alpha_1}$

where α_1 is an arbitrary exponent (typically $\alpha_1 \in [1,5]$).

We note : $y'_2 = dm/dx$ at $x = x_2$

"PBL" sub-domain : (for $x_3 < x \leq x_4$) :

We note $d_3 = \dfrac{(1-x_4)(1-y_3) - (1-x_3)(1-y_4)}{(1-x_4)}\,(x_4 - x_3)^{-\alpha_3}$

$m(x) = 1 - [(1-y_4)/(1-x_4)]\,(1-x) - d_3\,(x_4 - x)^{\alpha_3}$

We note : $y'_3 = dm/dx$ at $x = x_3$

where α_3 is an arbitrary exponent (typically $\alpha_3 \in [1,5]$).

bottom sub-domain : (for $x_4 < x \leq 1$) :

$m(x) = 1 - [(1-y_4)/(1-x_4)]\,(1-x)$ (used only if an analytic expression of A, B is needed)

"Tropo" sub-domain : (for $x_2 < x \leq x_3$) :

We note : $\Delta x = (x_3 - x_2)$, $\Delta y = (y_3 - y_2)$, $s = \Delta y/\Delta x$.

$m(x) = y_2 + (x - x_2)\,y'_2 + (x - x_2)^2\,[\Delta x(s - y'_2) + (x - x_3)(y'_2 + y'_3 - 2s)]\,/\Delta x^2$

This is nothing else than the differentiable fit between the characteristic points (x_2, y_2) and (x_3, y_3).

4 Definition of the hybridicity function

The hybridicity function h is build by using two characteristic points and checks the fulfilment of the constraint represented by (20). The vertical domain of the stretched coordinate y is divided into three sub-domains :

- pure terrain-following bottom-subdomain : may never be empty, must contain at least the surface point $y = 1$, may contain the whole domain $y \in [y_T, 1]$.
- pure pressure top-subdomain : may be empty, must be empty if $z_T \neq 0$, is non-empty only if $z_T = 0$
- hybrid intermediate-subdomain : must contain exactly the space left by the two previous layers, should be empty only if the whole domain is in pure terrain-following coordinate.

The two interfaces between these sub-domains are used as characteristic points which are defined by their stretched coordinate y, computed in practice from a level index :

- stretched level of the top of pure terrain-following layer : $y_\sigma = m[(L - N_\sigma)/L]$
- stretched level of the bottom of pure pressure layer : $y_\pi = m[(N_\pi)/L]$

where N_σ is the number of pure terrain-following levels *in addition to the mandatory surface one*, and N_π is the number of pure σ and π levels respectively.
The number of non-surface terrain-following levels must satisfy :

$$N_\sigma \in \{0, ..., L\} \qquad (21)$$

Choosing $N_\sigma = L$ means that the whole domain is described with a pure terrain-following coordinate, whilst $N_\sigma = 0$ means that the only pure terrain-following level is the surface one.

The number of pure pressure levels is set to zero if $z_T > 0$ because this means that the top boundary is not a pure pressure level. When $z_T = 0$, N_π is free (however taking a large value will violate (20) for quite large values of π_s). In this case, specifying $N_\pi = 0$ results is $y_\pi = m[0] = y_T$, that is the only pure pressure level is the top one (which is also in this case a pure terrain-following one).
The constraint (20) imposes :

$$y_\pi \leq y_\sigma \frac{X-1}{X} \tag{22}$$

where X is defined in (20).
When $y_T = 0$ the value of z_T may need to be enlarged in order to guarantee that (20) can actually be fulfilled :

$$z_T = \max[0, y_T - X(y_\sigma - y_T)] \tag{23}$$

The bounds y_σ, y_π, and z_T are now ready for a proper definition of the hybridicity function satisfying the constraint (20). We define :

$$d_1 = \alpha_h \frac{(y_\sigma - z_T)^2}{(y_\sigma - y_\pi)} \tag{24}$$

$$d_2 = 1 + \alpha_h \frac{(y_\sigma - z_T)}{(y_\sigma - y_\pi)} \tag{25}$$

where α_h is an arbitrary exponent (with typically $\alpha_h \in [-1, -3]$). The value $\alpha_h = -1$ results in a linear increase of h with y Then the hybridicity function h is defined by :

$h(y_T) = z_T$

$h(y) = 0$ for $y_T < y \leq y_\pi$ (this formula is active only when $y_T < y_\pi$).

$h(y) = z_T + \dfrac{d_1}{d_2 - [(y - y_\pi)/(y_\sigma - y_\pi)]^{\alpha_h}}$ for $y_\pi < y < y_\sigma$.

$h(y) = y$ for $y_\sigma \leq y \leq 1$.

It is seen that a pure σ coordinate is obtained by setting $N_\sigma = L$ and $N_\pi = 0$. When using actually hybrid coordinates in ARPEGE, it is a tradition to set $N_\sigma \geq 1$ and $N_\pi \geq 1$, i.e. there is at least two pure-pressure levels at top and at least two pure terrain-following levels at the bottom.

5 Free parameters

The free parameters for defining a vertical coordinate are then :

Physical parameters :
- L : total number of levels,
- N_{PBL} : number of levels in the "PBL",
- π_{PBL} : pressure of the "PBL" top,
- N_{Strato} : number of levels in the "stratosphere",
- π_{strato} : pressure of the "tropopause",
- $\delta\pi_L$: pressure depth of the bottom model layer,
- $\delta\pi_1$: pressure depth of the top model layer.
- N_σ : number of pure terrain following levels in addition to the mandatory surface one,
- N_π : number of pure pressure levels.
- y_T : pressure at the top of the model (where $\pi_s = \pi_{00}$), divided by π_{00}.
- z_T : hybridicity at top (this parameter can be overwritten by the program in order that (20) could be fulfilled.

Mathematical parameters :
- α_1 : exponent for the "stratosphere",
- α_3 : exponent for the "PBL",
- α_h : exponent for the hybridicity.

6 Tuning of mathematical parameters

The mathematical parameters α_1, α_3, α_h have no immediate meaning. It is enough to know that they define the rate at which the curves in the sub-domain depart from linear functions joining the characteristic points. For instance, $\alpha_1 = \alpha_3 = 1$ means linear functions, while e.g. $\alpha_1 = \alpha_3 = 5$ means high resolutions very concentrated near the edges of the domain. Similarly, $\alpha_h = -1$ means an hybridicity linearly increasing with the level index, and $\alpha_h = -3$ means a smoother variation. The impact of α_h is however quite weak.
Since the setting of the mathematical parameters α_1, α_3 is not fully intuitive, the program provides reasonable default values :

$$\alpha_1 = \left(\frac{0.8\, y_3 - y_2}{x_3 - x_2} - \frac{y_1(x_2 - x_1)}{x_1 y_2 - x_2 y_1} \right) \tag{26}$$

and :

$$\alpha_3 = \left(\frac{1.4\, y_2 - y_3}{x_2 - x_3} - \frac{(1 - y_4)(x_4 - x_3)}{(1 - x_4)(1 - y_3) - (1 - x_3)(1 - y_4)} \right) \tag{27}$$

For the parameter α_h, as outlined above, the impact is weak and values slightly smaller than -1 should always be a good choice. In any case $\alpha_h = -1$ leads to an effective coordinate, but the shape of the resulting hybridicity function is not smooth (differentiable). Differentiability is achieved for exponents smaller than -1, but with a larger risk of non-monotonous coordinate. In any case, it is recommended to plot the obtained coordinate in order to see if the result is aesthetically convenient.

7 Conclusions

In the future, it is anticipated that a local increase of resolution will be desired near the tropopause. Achieving this goal is not allowed by the proposed formulation, and would require the addition of two other characteristic points in the tropopause area in order to describe the mean resolution desired there. Another possibility, maybe simpler to implement, would be to add only one characteristic point (the "pressure" at the location of local maximum resolution), and a resolution at this characteristic point (i.e. the slope of the m function there).