Scientific Documentation

for Aladin-NH Dynamical Kernel

Pierre Bénard

CNRM/GMAP/MOD

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Chapter 1

General Remarks

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1.1 Introduction

Aladin-NH is a non-hydrostatic (NH) limited-area model (LAM) dedicated to numerical weather prediction (NWP). It is an extension of the Aladin model in order to include nonhydrostatic effects. The initial design of the dynamical kernel of the Aladin-NH model is based on the paper of Laprise (1992), advocating the mass-coordinate as a natural way for extending to the fully elastic system of Euler equations (EE) a pre-existing hydrostatic primitive equations (HPE) NWP system based on pressure coordinates.

The general program for the dynamical kernel of Aladin-NH was to built a system which is:

- Valid at any scale
- At least second-order accurate in time and space
- As efficient as possible
- Preserving as much as possible the global invariants

The strategy chosen in Aladin-NH to fulfil the latter general program is the following:

- Fully elastic set of Euler equations
- Mass-based vertical coordinate
- Spectral (bi-Fourier) transform method for horizontal directions
- Finite difference with a Lorenz grid in the vertical
- Constant-coefficients Semi-implicit (SI) or Iterative-centred-implicit (ICI) time-scheme.
- Semi-Lagrangian transport scheme.

The main reported limitations for the dynamical kernel of Aladin-NH are:

- "classical" approximation
- vertically unbounded atmosphere only
- monophasic set of equations

1.1.1 Euler Equations

The fully elastic set of Euler equations, is the basic set of the local fluid mechanics for compressible fluids.

This set of equation is non-hydrostatic because it does not retain the hydrostatic approximation. The hydrostatic approximation assumes that vertical accelerations are much smaller than the gravitational acceleration. As an indirect consequence, the vertical velocity is no longer a prognostic variable, i.e. there is a purely diagnostic link between the vertical velocity and the other dynamical fields. Systems retaining the hydrostatic assumption (such as e.g. HPE) are valid only for scales larger than a given limit, which is generally admitted to be the convective scale. Since it does not retain the hydrostatic approximation, the EE system has the advantage of being realistic even at small scales.

On the opposite side, the EE system is "non-anelastic" in the sense that it does not retain the anelastic approximation. In the anelastic approximation, the fluid is assumed to be formally incompressible. As an indirect consequence, the pressure is no longer a prognostic variable, i.e. there is a diagnostic link between the pressure and the other dynamical fields. Systems retaining the anelastic approximation are usually considered to be not valid for very large scale motions. Davies et al. (2003) argue that anelastic systems give an erroneous propagation for large-scale disturbances of meteorological importance, such as Rossby waves. Since it does not retain the anelastic approximation, the EE system has the advantage of being realistic even for large scales motions.

Therefore the EE set of equations has the advantage of being valid at any scale. Besides, all types of dynamical waves present in the atmosphere can be modelled with the EE system, including elastic waves (also called "acoustic" waves if their frequency is in the range of human's ear sensitivity). However, the possibility of elastic waves to be represented creates specific problems which must be solved if a numerically efficient kernel is desired. This point is developped below, in section 1.1.4.

1.1.2 Shallow-atmosphere and "classical" approximations

It is useful to first outline the difference between the spherical deep-atmosphere assumption and the shallow-atmosphere approximation. In a spherical deep-atmosphere, the vertical lines are not mutually parallel, but diverging with height. This is due to the assumed sphericity of the underlying planet. As a consequence, if we consider a vertical column which has a unit area at ground, the surface of this column increases (quadratically) with the distance r to the spherical planet's center. Moerover, the gravity decreases (quadratically) with r. The shallow-atmosphere approximation consists in neglecting these two effects, hence the area of a vertical column, and the gravity are assumed independent of height. As a consequence, the geometrical infinitesimal element has a constant area. The shallow-atmosphere approximation is well-adapted when one is interested in describing the evolution inside a thin layer above the ground. However, if one is interested by the evolution above 50-100 km (for the earth), the shallow-atmosphere approximation should be relaxed.

The Coriolis force has non-vanishing components in all directions in the general case. However, when the resolution is not very fine, it is usual to neglect the vertical component of the Coriolis force as well as the horizontal components resulting from the vertical velocity. This simplified-Coriolis approximation is well-adapted when vertical velocities remain smaller than the horizontal ones, but at small scales, this can be false and the total Coriolis force should be used.

The "traditional" approximation is the combination of the shallow-atmosphere and the simplified-Coriolis approximation. It is known (e.g. White and Bromley, 1995) that the combination of these two assumptions is more consistent than one of these taken alone. Hence relaxation of these approximations should be done in the same time.

1.1.3 mass-based coordinate

In small-scale meteorology, two main classes of vertical coordinates are possible: height-based and mass-based.

Pressure or temperature coordinates, which had been used in the past (at the time of large-scale meteorology) are not valid for small scale meteorology because they are not insured to be monotonic with height at small scales. For instance, with the hydrostatic approximation, the pressure is a monotonic function with height, but this is no longer true for non-hydrostatic systems. This point is easy to check in the EE system. The adiabatic vertical momentum equation writes in z coordinate:

$$\frac{dw}{dt} + g + \frac{RT}{p}\frac{\partial p}{\partial z} = 0$$

Thus, $(\partial p/\partial z)$ becomes negative as soon as $(dw/dt) = -g$. It can be checked that this situation occurs at ground even for relatively moderate winds, for a mountain of scale smaller than a kilometer and few hundreds of meters high. The pressure ceases to be increasing with height when the apparent gravitation in a framework linked to the fluid vanishes or becomes negative.

Height-based coordinates are based on the geometrical height, and they have been traditionally used for the design of non-hydrostatic models. However, Laprise (1992), showed that choosing the mass as a coordinate was leading to a system which had many similarities with the HPE system in pressure coordinate. Laprise claimed that starting from a pre-existing numerical HPE model in pressure coordinate, the transformation to obtain a EE model in hydrostatic-pressure coordinate required only very limited changes. Moreover, the very strong similarity between the two formulations was claimed to be an advantage for the comparison of results obtained with the HPE and EE versions. Since the original HPE Aladin model was in pressure coordinate, it was thus decided to adopt the strategy proposed by Laprise.

Here we must explain the difference between the mass-coordinate and the so-called hydrostatic-pressure coordinate proposed by Laprise. The difference between these two types of coordinates only appears in deep atmospheres. For deep-atmospheres, the mass above a ground area is determined by integrating along geometrical (diverging) vertical columns, while the hydrostatic pressure must be determined by integrating the mass in a *constant-area* vertical column. Physically, this may be viewed as a consequence of the upward-directed resulting force exerted by all the pressure forces acting on the diverging conic sides of a diverging column. Hence the hydrostatic-pressure is always smaller than the weight of the mass above.

For deep atmospheres, the mass-coordinate is therefore not equivalent to the hydrostatic-pressure coordinate.

Conversely, when the shallow-atmosphere approximation is retained (as in Laprise, 1992), the column-integrated mass and the hydrostatic pressure only differ by a constant factor, and then the two coordinates become equivalent. In other words the hydrostatic-pressure coordinate becomes a particular case of the mass coordinate when the shallow atmosphere approximation is retained.

Note: It has been shown by Wood and Staniforth (2003), that for the EE system, the natural extension of the hydrostatic-pressure coordinate from a shallow-atmosphere to a deep-atmosphere is actually the mass-coordinate, and not the hydrostatic-pressure coordinate.
It should also be noticed that in hydrostatic systems, the apparent gravitation in the fluid framework is necessarily positive, and the pressure is thus a perfectly legitimate coordinate. In the HPE system (which includes the shallow-atmosphere and the hydrostatic approximations), the pressure is equal to the hydrostatic-pressure by construction, and hence the pressure coordinate is nothing else that a mass-based coordinate.

In the present version of Aladin-NH, the shallow atmosphere approximation is made, hence the hydrostatic-pressure is used as a particular case of the mass-based coordinate.

1.1.4 SI and ICI schemes

In the EE system, the presence of elastic waves which travel at a very high speed in any direction, would impose a very stringent wave-CFL stability criterion (especially in the vertical direction, for which the resolution is very high) if the evolution was treated explicitly. With the previous generation of HPE models, the semi-implicit (SI) time-discretisation, combined with semi-Lagrangian (SL) transport schemes has demonstrated its ability to efficiently alleviate the wave-CFL limitations due to the propagation of fast waves. For fully elastic models SI schemes have already been implemented with success in several other centers. It was thus decided to adopt this strategy for Aladin-NH as well.

SI schemes are based on an arbitrary separation of the evolution terms between a linear part, treated in a centred-implicit way, and the so-called non-linear (NL) residuals, treated explicitly. Let assume that the evolution of the system writes:

$$\frac{\partial \mathcal{X}}{\partial t} = \mathcal{M}(\mathcal{X})$$

then the SI scheme writes:

$$\frac{\delta \mathcal{X}}{\delta t} = \mathcal{M}(\mathcal{X}) + \mathcal{L}(\overline{\mathcal{X}}^t - \mathcal{X})$$

where $\overline{\mathcal{X}}^t$ represents a temporal average of the state-variable $\mathcal{X}$ in the time period δt. The operator $\mathcal{L}$ is the linear operator used to define the implicitly-treated terms. Generally speaking, the $\mathcal{L}$ operator is arbitrary, but of course, if $\mathcal{L}$ is chosen in a wrong way, the SI scheme will not have the advantages which are expected from it in comparison to an explicit scheme. The most commonly used approach to try to insure the relevance of this linear separation method, is to seek $\mathcal{L}$ as the tangent-linear operator of the complete operator $\mathcal{M}$ around a reference state $\mathcal{X}^*$, or at least an approximation of this operator. It is shown in section 3.3.5 that is approach is not always optimal, and better linear systems can sometimes be found, based on a careful analysis of the problem.

The ICI scheme is an iteration of the SI scheme in order to converge toward a fully centred-implicit scheme which writes:

$$\frac{\delta \mathcal{X}}{\delta t} = \overline{\mathcal{M}(\mathcal{X})}^t$$

However, this would require the inversion of a non-linear implicit system, a task which is not possible with direct methods. Hence, an iterative algorithm based on a generalisation of the fixed-point algorithm, is used as follows:

$$\begin{aligned}
\overline{\mathcal{M}(\mathcal{X})}^{t\,(0)} &= \mathcal{M}(\mathcal{X}) \\
\left[\frac{\delta \mathcal{X}}{\delta t}\right]^{(i)} &= \overline{\mathcal{M}(\mathcal{X})}^{t\,(i-1)} + \mathcal{L}(\overline{\mathcal{X}}^{t\,(i)} - \overline{\mathcal{X}}^{t\,(i-1)})
\end{aligned}$$

where $\overline{\mathcal{X}}^{t\,(i)}$ is a time average in the period δt after the (i)-eth iteration. The principle of SI and ICI schemes is explained in more details in the chapter 5.

According to the nature of the implicitly-treated linear terms , three main types of SI schemes can be distinguished. The coefficients of the linear terms can be:

(i) Constant in time and horizontally

(ii) Constant in time but not horizontally

(iii) Non-constant

When the above common approach involving the linearization of $\mathcal{M}$ around a reference state $\mathcal{X}^*$ is chosen, these three classes of SI schemes can be distinguished according to the properties of $\mathcal{X}^*$:

(i) $\mathcal{X}^*$ is a stationnary and horizontally homogeneous state

(ii) $\mathcal{X}^*$ is a stationnary state, but non-homogeneous

(iii) $\mathcal{X}^*$ is non-stationnary and non-homogeneous.

The SI and ICI time-discretisations developed in Aladin-NH belong to the first class. The stationarity and the horizontal homogeneity of $\mathcal{X}^*$allow a very simple inversion of the implicit system in the spectral space. This strategy is thus very well adapted for a spectral model.

1.2 History of Aladin-NH dynamical core

- 1993: The development of the Aladin-NH dynamical core began in 1993, after the publication of Laprise's paper.
- 1994: First version of the dynamical kernel. This version was Eulerian only, and used the following nonhydrostatic prognostic variables:

$$\begin{aligned} \hat{\mathcal{P}} &= \frac{p-\pi}{\pi^*} \\ \hat{d} &= -g\frac{\pi^*}{m^* R T^*}\left(\frac{\partial w}{\partial \eta}\right). \end{aligned}$$

  An iteration of the so-called "X-term" (cf: section 2.3.8 for its definition) was necessary to avoid some unexplained severe instabilities. An "heuristic"(i.e. ad hoc) discretization of the horizontal wind-shear near the top and bottom boundaries was also necessary to maintain the stability of the model (cf last paragraph of section 6 of Bunbnova at al., 1995).
- 1995: The extension from the Eulerian to a three-time-level (3-TL) semi-Lagrangian (SL) scheme fails, due to instabilities at large time-steps.
- 1996: The implementation of a two-time-levels (2-TL) SL scheme leads to a very severe instability of the system. The model holds only for a few time-steps before blowing-up.
- 1996-1998: All developments are stopped, partly for political reasons (not to conflict with the emerging meso-NH research project and model) and partly due to a loss of motivation, facing the poor quality of the model stability.
- 1998: The theoretical investigation on the model behaviour is recognized as a research topic, and it is decided to re-open the subject with dedicated forces. For this, a 2D vertical plane version of the model is built to allow the academic research.
- 1999: An extensive diagnostic of the model weaknesses is undertaken. The instabilities are reproduced in Aladin-NH for more and more simple flows, for which a theoretical analysis is likely to be attainable. Besides, a significant effort is devoted to built theoretical tools for studying analytically the behaviour of the model. These tools mainly consist in 1D and pseudo 2D (for a single horizontal mode) numerical analyses of the complex numerical frequency of the model in academic conditions. Finally, the implementation of an iterative centred implicit (ICI) scheme is decided, to enforce the overall stability of the scheme.
- 2000: Two first weaknesses are solved: a mismatch between the linear and non-linear operator due to the specification of the conservation of angular momentum (see section 4.11.1); and an incorrect choice of the two non-hydrostatic prognostic variables. Two new variables are proposed: $\mathcal{P}$ and d (see section 2.3.7). As a consequence of these two points, the "heuristic" discretization of the wind at top and bottom (see item 1994) can be relaxed.
- 2001: Another weakness is identified: the above choice d is not enough to stabilize the model in presence of orography. A new variable $\mathbb{d}$ is proposed. The 3-TL SI SL scheme becomes stable and accurate for real cases at 2 km resolution.
- 2002: The weakness which prevents the use of 2-TL schemes is identified. A slight modification (called `SITRA`) allows to remove this obstacle (see section 3.3.5). The 2-TL SI SL scheme becomes stable and accurate for real cases at 2 km resolution. The ICI scheme is shown to allow an increase of stability in case of a very steep orography. The Aladin-NH model is chosen as the dynamical kernel of the future operational NH forecast system AROME.

- 2003: The bottom-boundary condition of the semi-Lagrangian version is improved (scheme `LNEWBC`, see section 4.7.2). This removes the spurious so-called "chimney effect" which was observed in orographic flows with the semi-Lagrangian version. Similarly, the horizontal-diffusion source term is introduced in the rigid bottom boundary condition of the vertical velocity to avoid a spurious stationary response for highly non-linear orographic flows.

- 2003-2004: A collaboration with the HIRLAM group is agreed in order that the Aladin-NH dynamical core can be used as a component of the NWP system of HIRLAM applications. For this, an additional geometry (rotated/tilted Mercator projection) has been introduced in Aladin. This geometry is conformal (being a Mercator projection) but is asymptotically close to the HIRLAM rotated lat/lon geometry for small domains. The description of the aladin geometry is not specific to the NH version, and is hence described in a separate document.

- 2004-2005: An harmonization of the techniques used in ARPEGE/IFS and Aladin to solve the discrete Helmholtz equation of the SI scheme is found desirable for the extension of the Aladin-NH dynamical kernel to global modelling: in ARPEGE/IFS the discrete Helmholtz operator is diagonalised and then inverted in its own vertical eigenspace, as required for solving properly the SI scheme in stretched applications (see Yessad and Bénard, 1996). In Aladin, the Helmholtz operator was directly inverted as a whole in the setup part, but this method is not valid for the global (possibly stretched) model. Therefore the technique used in ARPEGE/IFS is implemented in Aladin as well, in order to allow a transparent extension of the NH functionality to the global model. This ingredient is described in this version of the documentation, although not yet implemented in the code (see section 5.10).

- 2004-2005: It is anticipated that the Aladin-NH model could become unstable for larger domains than those used currently, because of the significant deviation of the actual map-factor $\mathbf{m}$ with respect to the linearized map-factor $\mathbf{m}_*$ in the SI scheme. A solution for this potential problem would be to use a spatially-variable linearized map-factor (as in the stretched version of ARPEGE). In this case, the algebraic manipulations to build the SI Helmholtz structure equation must take into account the non-commutativity of the map-factor multiplication operator with the horizontal derivative operator. This ingredients is described in this version of the documentation, although not yet implemented in the code (see sections 3.3.6, 3.4 and 5.10).

1.3 Versions of this documentation

Here is a list of the successive versions of this documentation.

- Version 1 (1995 $\cdots$ 2001) :

  The first releases of this documentation were not labelled as versions. They basically contained the description of the model corresponding to Bubnová et al., 1995:

  - Eulerian version only.
  - variables $\hat{\mathcal{P}}$ and $\hat{d}$
  - partial SI iteration of the so-called "X-term"
  - Eulerian bottom BC for the vertical momentum equation
  - inconsistency in angular momentum conservation constraint.
  - Necessary "heuristic" top- and bottom-boundary condition for the horizontal wind

- Version 2.0 (December 2003) :

  Contains the following modifications:

  - valid for Eulerian, SL 3-TL, and SL 2-TL versions
  - variables $\mathcal{P}$, d, and dl
  - SI and ICI time-discretisations
  - Eulerian and Lagrangian bottom BC for the vertical momentum equation
  - removed inconsistency in angular momentum conservation constraint.
  - removed "heuristic" top- and bottom-boundary condition for the horizontal wind
  - `SITRA` scheme for the 2-TL SL version.

- Version 2.1 (November 2004) :

  The present version. Contains the following modifications:

  - valid for the couples of variables $(\mathcal{P}, \mathrm{d})$, and $(\hat{q}, \mathrm{dl})$
  - Factorisation of the discrete SI Helmholtz operator as in ARPEGE/IFS (i.e. in the discrete vertical eigenmodes space, cf. section 5.10.1).
  - valid for spatially-variable linearized map-factor $\mathbf{m}_*$ in the SI scheme

Chapter 2

Continuous Equations in the Cartesian System

(18/11/2003)

2.1 Introduction

Mass-based coordinates, as proposed by Laprise (1992), are an attractive alternative to height-based coordinates for solving the Euler equations (EE) system. Moreover, when an orography is present, the introduction of a mass-based hybrid terrain-following coordinate "η" takes a very similar form as for the HPE system in the usual pressure-based coordinates.

- In a first step, the system of continuous compressible equations in the cartesian geometry is introduced in the natural set of prognostic variables (w, p) for the pure π coordinate and for the hybrid η coordinate.
- Then, the continuous sytem is derived for the specific set of prognostic variables used in Aladin-NH. The reason for these changes of prognostic variables is for increasing the robustness of the SI scheme, and to avoid vertical staggering (see section 2.3.7).
- The tangent-linear version of the continuous system around a steady and horizontally homogeneous basic state is presented, in order allow the derivation of the continuous structure equation and of the normal modes. This linear system will also serves as a basis for the design of the SI and ICI time-discretisations.
- The energy and angular-momentum global invariants are finally presented

2.2 Continuous Euler Equations in pure mass-based coordinates

The space and time continuous EE system for a cartesian geometry and a shallow atmosphere in mass-based coordinates is introduced here. The pure mass-based coordinate, which thus reduces to the pure hydrostatic-pressure coordinate π is first introduced, then the system is transformed into the hybrid hydrostatic-pressure terrain-following coordinate η, in an exactely similar way as for the HPE ARPEGE/IFS system (Simmons and Burridge, 1981). Finally, the system is derived for the two sets of NH prognostic variables used in Aladin-NH, that is $(\mathcal{P}, \mathsf{d})$ and $(\mathcal{P}, \mathbb{d})$.

2.2.1 The vertical coordinate "hydrostatic pressure", π

The hydrostatic-pressure π is defined at each point by the weight of the unit-area air column above this point. It is seen therefore that the π coordinate is a mass-coordinate due to the shallow-atmosphere assumption.
The first remark is that this coordinate is, *by construction*, monotonic with respect to the geometrical height, which is not the case for the pressure itself, as stated above.
To illustrate the difference between the pressure p and the hydrostatic-pressure π in a physical way, it is convenient to imagine what would happen in the case of a very rapid flow (let's say in the sonic range) in the area located just above a montaneous ridge. The local pressure is certainly dramatically weak in this area, due to the dynamical depressurisation, analogous to the one which occurs at the vicinity of a plane's wing extrados. At the opposite, the hydrostatic pressure in the area will be determinated only by the weight of the atmosphere in a unit-area column above the area, i.e. by the repartition of density in the regions above the one considered here. This hydrostatic-pressure thus has no special reason to be very small in the considered area.
In the real atmosphere, manometers and barometers give a measurement of the true pressure itself. Conversely to the true pressure, the hydrostatic-pressure is not accessible to a direct measurement with any local intrument. The knowledge of the hydrostatic pressure requires a knowledge of the density profile in the whole column above the considered point.
The π coordinate is defined in the local geographical (x, y, z) cartesian geometrical coordinate system by:

$$\frac{\partial \pi}{\partial z} = -\rho g$$

where ρ is the density. Hence:

$$\pi(x, y, z, t) = \int_z^{+\infty} \rho(x, y, z', t) g dz'$$

where Ox is oriented toward the east, Oy toward the north, and Oz upwards.

2.2.2 Domain limits in π coordinate

The atmosphere is assumed to be limited by:

$$\pi(x, y, z, t) \in [\pi_T(x, y, t), \pi_s(x, y, t)]$$

where π_T and π_s are two functions depending on (x, y, t) only. However, it can be specified without loss of generality that π_T is a pure constant (see Chapter 9). As a consequence, the domain limits are defined by:

$$\pi(x, y, z, t) \in [\pi_T, \pi_s(x, y, t)] \tag{2.1}$$

Material boundary conditions

The limits of the domain are now specified to be material. This expresses the fact that the mass flux is zero through the limits of the domain. This writes in π coordinate (see section 9.3.2):

$$\dot{\pi}_{(\pi=\pi_s)} = \frac{\partial \pi_s}{\partial t} + V_s.\nabla \pi_s \tag{2.2}$$

$$\dot{\pi}_{(\pi=\pi_T)} = \frac{\partial \pi_T}{\partial t} + V_T.\nabla \pi_T \tag{2.3}$$

2.2.3 Euler equations system in π coordinate

The system describing the flow can be obtained from the classical height coordinate one by simply expressing the local geometry of the coordinate change:

$$\frac{\partial}{\partial z} = -\frac{p}{RT}g\frac{\partial}{\partial \pi}$$
$$\nabla_z = \nabla_\pi + \frac{p}{RT}(\nabla_\pi \phi)\frac{\partial}{\partial \pi}$$

where ϕ is the geopotential ($\phi = gz$).
The π-coordinate system then writes, in the local frame (see Laprise, 1992):

$$\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla_\pi p + \frac{\partial p}{\partial \pi}\nabla_\pi \phi = \mathcal{V} \tag{2.4}$$
$$\frac{dw}{dt} + g\left(1 - \frac{\partial p}{\partial \pi}\right) = \mathcal{W} \tag{2.5}$$
$$\nabla_\pi \mathbf{V} + \frac{\partial \dot{\pi}}{\partial \pi} = 0 \tag{2.6}$$
$$\frac{dT}{dt} - \frac{RT}{C_p}\frac{1}{p}\frac{dp}{dt} = \frac{Q}{C_p} \tag{2.7}$$
$$\frac{dp}{dt} + \frac{C_p}{C_v}pD_3 = \frac{Qp}{C_v T} \tag{2.8}$$
$$\frac{d\phi}{dt} = gw \tag{2.9}$$
$$\frac{\partial \phi}{\partial \pi} = -\frac{RT}{p} \tag{2.10}$$

with the following notations:

- $\mathbf{V}$: Horizontal wind vector.
- ∇_π : Horizontal gradient on constant π surfaces.
- $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla_\pi + \dot{\pi}\frac{\partial}{\partial \pi}$: Lagrangian derivative.
- p : *True* pressure.
- T : Temperature.
- w : Vertical velocity (dz/dt).
- $D_3 = \nabla_\pi.\mathbf{V} + \rho(\nabla_\pi \phi).\left(\frac{\partial \mathbf{V}}{\partial \pi}\right) - g\rho\frac{\partial w}{\partial \pi}$: True 3-dimensional divergence of the wind.
- $\rho = \frac{p}{RT}$: density .
- $\mathcal{V}, \mathcal{W}, Q$: Physical components of the forcing ($\mathcal{V}$ includes Coriolis term).

2.3 Continuous Euler Equations in hybrid mass-based coordinates

2.3.1 The hybrid vertical coordinate η

The hybrid coordinate η makes simpler the representation of dynamically consistent bottom boundary conditions in presence of orography. This coordinate can be introduced from π in exactely the same way as the pressure hybrid coordinate is usually introduced from the pressure coordinate in hydrostatic models:

$$\pi(x, y, \eta, t) = A(\eta) + B(\eta)\pi_s(x, y, t)$$

where π_s is the ground hydrostatic pressure (i.e. the weight of a unit-area air column above the ground).

NOTATION: We will note henceforth $m(x, y, \eta, t)$ the vertical metric factor:

$$m = \frac{\partial \pi}{\partial \eta} \tag{2.11}$$

The variation domain of η can be defined by $[\eta_s, \eta_T]$. We impose, without any loss of generality, the numerical value of the η coordinate at the two boundaries to be:

$$\eta_s = 1 \tag{2.12}$$

$$\eta_T = 0 \tag{2.13}$$

The functions A and B are two arbitrary functions. Indeed they are not totally arbitrary because they must satisfy:

$$\frac{dA}{d\eta} + \pi_s \frac{dB}{d\eta} > 0$$

for any value of π_s in the domain. In practice, A and B are chosen in such a way that:

$$\frac{dA}{d\eta} + \pi_{sMin} \frac{dB}{d\eta} > 0$$

where π_{sMin} is a number smaller that the smallest likeky value of π_s in the domain and through all the duration of the forecast.

2.3.2 Upper domain limit

The upper limit of the domain specified in η is given by $\eta = 0$ and $\pi = \pi_T$. As mentionned above π_T is a pure constant. Hence the upper boundary limit thus writes in terms of A and B:

$$A(0) = \pi_T$$
$$B(0) = 0$$

In a first stage (and in the present version of the documentation) it is assumed that $\pi_T = 0$. This choice is well-suited to describe a vertically unbounded atmosphere (see Chapter 9). Hence for the current version of the documentation:

$$A(0) = 0$$
$$B(0) = 0$$

However, since ARPEGE/Aladin is considered as a community tool with possible meso-scale applications, it is planned to implement the possibility to run the model with a vertically bounded atmosphere. In this case, $\pi_T = \text{Cst} \neq 0$ has to be imposed as the upper limit of the domain (see also Chapter 9)

2.3.3 Lower domain limit

The lower limit of the domain is specified by $\eta = 1$ and $\pi = \pi_s(x, y, t)$, which in terms of A and B functions writes:

$$\begin{aligned} A(1) &= 0 \\ B(1) &= 1 \end{aligned}$$

2.3.4 Material boundary conditions

The specification of material boundary conditions into the η coordinate yields (see section 9.3.2):

$$\dot{\eta}(1) = 0 \quad (2.14)$$

$$\dot{\eta}(0) = 0 \quad (2.15)$$

Note: As in ARPEGE (so-called case "$\delta m = 1$"), the conservation of the atmospheric air total mass during evaporation/precipitation processes at ground level can be handled trough a specification of $\dot{\eta}(1)$. In a similar way, the equations of the AROME model for the multi-phasic atmosphere are planned to take into account the mass-flux of the total atmospheric parcel across the ground surface. However, these aspects are not discussed in the current version of the documentation.

2.3.5 Transformation rules

The transformation rules from π toward η are:

$$\frac{\partial}{\partial \pi} = \frac{1}{m}\frac{\partial}{\partial \eta} \quad (2.16)$$

$$\nabla_\pi = \nabla_\eta - (\nabla_\eta \pi)\frac{1}{m}\frac{\partial}{\partial \eta} \quad (2.17)$$

Which yields:

$$m = \frac{dA}{d\eta} + \pi_s \frac{dB}{d\eta} \quad (2.18)$$

$$\nabla_\eta \pi = B\nabla_\eta \pi_s = B\pi_s \nabla(\ln \pi_s) \quad (2.19)$$

2.3.6 Euler equations system in hybrid coordinates η

The system cast in π coordinates can easily be transformed to η by application of the above transformation rules between the two systems:

$$\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla \phi = \mathcal{V} \quad (2.20)$$

$$\gamma \frac{dw}{dt} + g\left(1 - \frac{1}{m}\frac{\partial p}{\partial \eta}\right) = \gamma \mathcal{W} \quad (2.21)$$

$$\frac{dT}{dt} - \frac{RT}{C_p}.\frac{1}{p}\frac{dp}{dt} = \frac{Q}{C_p} \quad (2.22)$$

$$\frac{dp}{dt}+\frac{C_p}{C_v}pD_3 = \frac{Qp}{C_vT} \tag{2.23}$$

$$\frac{\partial m}{\partial t}+\nabla(m\mathbf{V})+\frac{\partial}{\partial\eta}(m\dot{\eta}) = 0 \tag{2.24}$$

$$\frac{d\phi}{dt} = gw \tag{2.25}$$

$$\frac{\partial\phi}{\partial\eta} = -m\frac{RT}{p} \tag{2.26}$$

where ∇ is the horizontal derivative operator along constant η surfaces (noted ∇_η above). The expression for the local 3-dimensional divergence D_3 is:

$$D_3=\nabla.\mathbf{V}+\frac{1}{m}\frac{p}{RT}\nabla\phi.\left(\frac{\partial\mathbf{V}}{\partial\eta}\right)-\frac{g}{m}\frac{p}{RT}\frac{\partial w}{\partial\eta} \tag{2.27}$$

The integration of the continuity equation on the vertical through the whole depth of the atmosphere leads to the surface hydrostatic pressure tendency equation:

$$\frac{\partial\pi_s}{\partial t}+\nabla\int_0^1 m\mathbf{V}d\eta=0 \tag{2.28}$$

In the same way, integrating from the top to the current level gives the pseudo-vertical velocity in η coordinate:

$$m\dot{\eta}=B\int_0^1\nabla.m\mathbf{V}d\eta-\int_0^\eta\nabla.m\mathbf{V}d\eta$$

Finally the Lagrangian derivative of hydrostatic pressure $\dot{\pi}$ (classically named ω) can be obtained from the two previous equations, and leads to the following diagnostic relation:

$$\omega=\dot{\pi}=\mathbf{V}.\nabla\pi-\int_0^\eta\nabla.m\mathbf{V}d\eta \tag{2.29}$$

The horizontal gradient of geopotential is given by vertically integrating (2.26):

$$\nabla\phi=\nabla\phi_s+\int_\eta^1\nabla\left(\frac{mRT}{p}\right)d\eta \tag{2.30}$$

2.3.7 Formulation with reduced non-hydrostatic pressure departure "$\mathcal{P}$" and vertical divergence "d"

Traditionnally, the problem of big cancelling terms in vertical momentum equation is alleviated by using the non-hydrostatic pressure departure $p'=p-\pi$ as a prognostic variable instead of the true pressure p. To avoid some instability in the semi-implicit scheme (see section 5.7), this departure p' is rescaled by the hydrostatic pressure π. Finally, the NH prognostic variable for the pressure equation is:

$$\mathcal{P}=\frac{p-\pi}{\pi}$$

The prognostic equation for $\mathcal{P}$ then is obtained by logarithmic derivative of its definition:

$$\begin{aligned}\frac{d\mathcal{P}}{dt} &= (1+\mathcal{P})\left(\frac{1}{p}\frac{dp}{dt}-\frac{1}{\pi}\frac{d\pi}{dt}\right)\\ &= -(1+\mathcal{P})\left(\frac{C_p}{C_v}D_3+\frac{\dot{\pi}}{\pi}\right)+(1+\mathcal{P})\frac{Q}{C_vT}\end{aligned} \tag{2.31}$$

where the diagnostic fields D_3 and $(\dot{\pi}/\pi)$ are obtained from (2.27) and (2.29).

In a similar way, the stability of the semi-implicit scheme calls for a change towards a pseudo vertical divergence:

$$\mathsf{d} = -g\frac{\rho}{m}\frac{\partial w}{\partial \eta} \tag{2.32}$$

in replacement of the original vertical velocity variable. The density is given by $\rho = [\pi(1+\mathcal{P})/RT]$.
Additionally this choice of a new prognostic variable in term of divergence avoids to have a vertical staggering of the vertical momentum prognostic variables, as it would be the case if the vertical velocity w was chosen as a prognostic variable. This point is important because a vertical staggering of prognostic variables would automatically imply the necessity of having two separate sets of origin points in the semi-Lagrangian scheme (one set for the non-staggered variables and one set for the vertically staggered variables). For the simplicity (and efficiency) of the semi-lagrangian model, the absence of vertical staggering is thus a real advantage. This naturally lead to use d in lieu of w as a prognostic variable.
The vertical velocity is diagnosed from d through:

$$w = w_s + \int_{\eta}^{1} \nabla\left(\frac{mRT\mathsf{d}}{g\pi(1+\mathcal{P})}\right) d\eta \tag{2.33}$$

From the logarithmic derivation of the above definition, one obtains the following evolution equation for d:

$$\frac{d\mathsf{d}}{dt} = \mathsf{d}\frac{1}{p}\frac{dp}{dt} - \mathsf{d}\frac{1}{T}\frac{dT}{dt} - \mathsf{d}\frac{1}{m}\frac{dm}{dt} - g\frac{p}{mRT}\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right)$$

one can write:

$$\frac{1}{T}\frac{dT}{dt} - \frac{1}{p}\frac{dp}{dt} = D_3$$

Moreover we have:

$$\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right) = \frac{\partial \dot{w}}{\partial \eta} - \frac{\partial \mathbf{V}}{\partial \eta}.\nabla w - \frac{\partial \dot{\eta}}{\partial \eta}.\frac{\partial w}{\partial \eta}$$

which yields:

$$\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right) = g\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial \pi\mathcal{P}}{\partial \eta}\right) - \frac{\partial \mathbf{V}}{\partial \eta}.\nabla w - \frac{\partial \dot{\eta}}{\partial \eta}.\frac{\partial w}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial \eta}$$

and we also have:

$$\frac{1}{m}\frac{dm}{dt} = -\left(\nabla.\mathbf{V} + \frac{\partial \dot{\eta}}{\partial \eta}\right)$$

Hence:

$$-\frac{\mathsf{d}}{m}\frac{dm}{dt} - g\frac{p}{mRT}\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right) = \mathsf{d}D - g^2\frac{p}{mRT}\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial \pi\mathcal{P}}{\partial \eta}\right) - g\frac{p}{mRT}\frac{\partial \mathcal{W}}{\partial \eta} + g\frac{p}{mRT}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w$$

and finally, the prognostic equation for d writes:

$$\frac{d\mathsf{d}}{dt} = -g^2\frac{p}{mRT}\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial \pi\mathcal{P}}{\partial \eta}\right) + g\frac{p}{mRT}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w + \mathsf{d}(\nabla.\mathbf{V} - D_3) - g\frac{p}{mRT}\frac{\partial \mathcal{W}}{\partial \eta} \tag{2.34}$$

The quantity ∇w is obtained diagnostically from (2.33):

$$g\nabla w = g\nabla w_s + \int_{\eta}^{1} \frac{mRT}{p}\nabla\mathsf{d}\, d\eta' - \int_{\eta}^{1} \mathsf{d}\,\frac{\partial \nabla\phi}{\partial \eta} d\eta'$$

Using the new variables and combining the temperature and pressure equation, the original η-system can thus be rewritten:

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla\phi &= \mathcal{V} && (2.35)\\
\frac{d\mathsf{d}}{dt} + g^2\frac{p}{mRT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial \pi\mathcal{P}}{\partial\eta}\right) - g\frac{p}{mRT}\frac{\partial \mathbf{V}}{\partial\eta}.\nabla w - \mathsf{d}(\nabla.\mathbf{V} - D_3) &= -g\frac{p}{mRT}\frac{\partial\mathcal{W}}{\partial\eta} && (2.36)\\
\frac{dT}{dt} + \frac{RT}{C_v}D_3 &= \frac{Q}{C_v} && (2.37)\\
\frac{d\mathcal{P}}{dt} + (1+\mathcal{P})\frac{C_p}{C_v}D_3 + (1+\mathcal{P})\frac{\dot{\pi}}{\pi} &= (1+\mathcal{P})\frac{Q}{C_vT} && (2.38)\\
D_3 = \nabla.\mathbf{V} + \mathsf{d} &+ \frac{p}{mRT}\nabla\phi\,.\left(\frac{\partial\mathbf{V}}{\partial\eta}\right) && (2.39)\\
\frac{\partial \pi_s}{\partial t} + \int_0^1 \nabla\,(m\mathbf{V})d\eta &= 0 && (2.40)\\
\frac{d\phi}{dt} &= gw && (2.41)\\
\frac{\partial\phi}{\partial\eta} &= -m\frac{RT}{\pi(1+\mathcal{P})} && (2.42)\\
p &= \pi(1+\mathcal{P}) && (2.43)
\end{aligned}
$$

2.3.8 Formulation with reduced non-hydrostatic pressure departure "$\hat{q}$" and modified vertical divergence "$\mathbb{d}$"

In this section we introduce two alternative prognostic nonhydrostatic variables ($\hat{q}$, $\mathbb{d}$) in replacement of ($\mathcal{P}$, d). The replacement of $\mathcal{P}$ by $\hat{q}$ brings no significant differences in terms of the performance of the model, as far as experiments have shown it, but this latter variable $\hat{q}$ has been elected as the most probable variable to be kept when the code will be cleaned in order to eliminate redundant possibilities.This new variable $\hat{q}$ is defined by:

$$\hat{q} = \ln(p/\pi) \tag{2.44}$$

On the other hand, the variable $\mathbb{d}$ leads to a more robust scheme in presence of orography than the variable d (Bénard et al., 2004b) . This new variable $\mathbb{d}$ is defined by:

$$\mathbb{d} = \mathsf{d} + \frac{p}{mRT}\nabla\phi\,.\left(\frac{\partial\mathbf{V}}{\partial\eta}\right) \equiv D_3 - \nabla\mathbf{V} \tag{2.45}$$

The cross-term $\mathbb{d} - \mathsf{d} = D_3 - \nabla\mathbf{V}$ is traditionally called the "X-term" and will be noted X:

$$\mathsf{X} = \frac{p}{mRT}\nabla\phi\,.\left(\frac{\partial\mathbf{V}}{\partial\eta}\right) \tag{2.46}$$

The evolution equation for $\mathbb{d}$ can be derived from (2.34):

$$
\begin{aligned}
\frac{d\mathbb{d}}{dt} &= -g^2\frac{p}{mRT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial\pi(\exp\hat{q}-1))}{\partial\eta}\right) + g\frac{p}{mRT}\frac{\partial\mathbf{V}}{\partial\eta}.\nabla w\\
&\quad +\mathbb{d}(\mathsf{X}-\mathbb{d}) - g\frac{p}{mRT}\frac{\partial\mathcal{W}}{\partial\eta} + \dot{\mathsf{X}} && (2.47)
\end{aligned}
$$

The last term is left in the form of a lagrangian derivative $\dot{\mathsf{X}}$ on purpose, because it will have a specific time-discretisation (see section 5.8). With this new variable, the 3-D divergence writes:

$$D_3 = \nabla \mathbf{V} + \mathbb{d} \tag{2.48}$$

The other equations are not formally modified with respect to (2.35)–(2.43), except that the expression of D_3 to be used is (2.48) instead of (2.39).
The evolution equation for $\hat{q}$ can be derived from (2.31):

$$\frac{d\hat{q}}{dt} = -\left(\frac{C_p}{C_v} D_3 + \frac{\dot{\pi}}{\pi}\right) + \frac{Q}{C_v T} \tag{2.49}$$

2.4 Conservation of Energy

It has been shown by Laprise (1992) that the total energy invariant on the sphere is defined, for the considered system by:

$$\int_0^{2\pi} \int_{-\pi/2}^{-\pi/2} \int_0^1 (K + C_v T + \phi) m.d\eta.a \cos\varphi.d\varphi.d\lambda \tag{2.50}$$

where λ and φ are the longitude and latitude, and K is the 3-dimensional kinetic energy: $K = (1/2)(\mathbf{V}^2 + w^2)$. The demonstration is not repeated here, and the reader is referred to Laprise, 1992 for more details. It is admitted here that $(K + C_v T + \phi)$ is the energy invariant also valid for the Cartesian framework used here.

2.5 Conservation of Total Angular Momentum

In an hydrostatic atmosphere, the conservation of angular momentum in the absence of external forcing can be expressed on a latitude circle (Simmons and Burridge, 1981) by:

$$\int_0^{2\pi} \left[\int_0^1 \left(\frac{\partial \phi}{\partial \lambda} + \frac{RT}{\pi} \frac{\partial \pi}{\partial \lambda} \right) m.d\eta + \phi_s \frac{\partial \pi_s}{\partial \lambda} \right] d\lambda = 0 \tag{2.51}$$

In the case of fully compressible fluid, this conservation law must be revised. Here we derive a mathematical rule which expresses a relationship between the geopotential and pressure gradients from the state equation, then we show how this rule contains the generalisation of (2.51) to the Euler equations system.

We have:

$$\frac{mRT}{p} = -\frac{\partial \phi}{\partial \eta}$$

Hence:

$$\begin{aligned}
\int_0^1 \frac{mRT}{p} \vec{\nabla} p \, d\eta &= -\int_0^1 \frac{\partial \phi}{\partial \eta} \vec{\nabla} p \, d\eta = -\int_0^1 \frac{\partial}{\partial \eta} (\phi - \phi_s) \vec{\nabla} p \, d\eta \qquad \left(\frac{\partial \phi_s}{\partial \eta} = 0 \right) \\
&= -\underbrace{\Big[(\phi - \phi_s) \vec{\nabla} p \Big]_0^1}_{0} + \int_0^1 (\phi - \phi_s) \frac{\partial}{\partial \eta} \vec{\nabla} p \, d\eta \qquad \left(p|_{\eta=0} = 0 \Longrightarrow \vec{\nabla} p|_{\eta=0} = 0 \right) \\
&= \int_0^1 (\phi - \phi_s) \vec{\nabla} \frac{\partial p}{\partial \eta} \, d\eta
\end{aligned}$$

$\Rightarrow$

$$\int_0^1 \frac{mRT}{p} \vec{\nabla} p \, d\eta = \int_0^1 (\phi - \phi_s) \vec{\nabla} \frac{\partial p}{\partial \eta} \, d\eta \tag{2.52}$$

Integrating along a closed latitude circle yields:

$$
\begin{aligned}
\int_0^{2\pi}\int_0^1 \frac{mRT}{p}\frac{\partial p}{\partial \lambda}\,d\eta\,d\lambda &= \int_0^{2\pi}\int_0^1 (\phi-\phi_s)\frac{\partial}{\partial\lambda}\left(\frac{\partial p}{\partial\eta}\right)\,d\eta\,d\lambda = \\
&= \int_0^1\int_0^{2\pi} \phi\frac{\partial}{\partial\lambda}\left(\frac{\partial p}{\partial\eta}\right)\,d\lambda\,d\eta - \int_0^{2\pi}\phi_s\int_0^1 \frac{\partial}{\partial\eta}\left(\frac{\partial p}{\partial\lambda}\right)\,d\eta\,d\lambda = \\
&= \int_0^1\left(\underbrace{\left[\phi\frac{\partial p}{\partial\eta}\right]_{\lambda=0}^{\lambda=2\pi}}_{0} - \int_0^{2\pi}\frac{\partial\phi}{\partial\lambda}\frac{\partial p}{\partial\eta}\,d\lambda\right)d\eta - \int_0^{2\pi}\phi_s\frac{\partial p_s}{\partial\lambda}\,d\lambda = \\
&= -\int_0^{2\pi}\int_0^1 \frac{\partial\phi}{\partial\lambda}\frac{\partial p}{\partial\eta}\,d\eta\,d\lambda - \int_0^{2\pi}\phi_s\frac{\partial p_s}{\partial\lambda}\,d\lambda
\end{aligned}
$$

which leads to the generalisation of (2.51) for the Euler equations system:

$$
\int_0^{2\pi}\left[\int_0^1\left(\frac{\partial\phi}{\partial\lambda}\frac{\partial p}{\partial\eta} + \frac{RT}{p}\frac{\partial p}{\partial\lambda}\frac{\partial\pi}{\partial\eta}\right)d\eta + \phi_s\frac{\partial p_s}{\partial\lambda}\right]d\lambda = 0 \tag{2.53}
$$

The connection between (2.53) and angular momentum conservation is not yet very clear at this point. Further analysis shows that this formula (multiplied by $\frac{a^2}{g}\cos\varphi$ and integrated through φ) reflects the following requirement (applied on a frictionless atmosphere, and considering only the component parallel to earth's rotation axis):

> The net torque of *all* forces acting on system is equal to the net torque of *external* forces. In other words, the net torque of *internal* forces is zero, and the total angular momentum along the earth rotation axis is conserved in the absence of external forces.

It is seen that the conservation of the global angular-momentum follows indirectly from a stronger local relationship (2.52) for each column of the atmosphere. This local relationship will be used as a constraint for the design of the vertical discretisation.

2.6 Final Form of Dynamical Model Equations (variables $\mathcal{P}$, d)

2.6.1 Prognostic equations

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla\phi &= \mathcal{V} && (2.54)\\
\frac{d\mathsf{d}}{dt} + g^2\frac{p}{mRT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial \pi\mathcal{P}}{\partial\eta}\right) - g\frac{p}{mRT}\frac{\partial\mathbf{V}}{\partial\eta}.\nabla w - \mathsf{d}(\nabla.\mathbf{V} - D_3) &= -g\frac{p}{mRT}\frac{\partial\mathcal{W}}{\partial\eta} && (2.55)\\
\frac{dT}{dt} + \frac{RT}{C_v}D_3 &= \frac{Q}{C_v} && (2.56)\\
\frac{d\mathcal{P}}{dt} + (1+\mathcal{P})\frac{C_p}{C_v}D_3 + (1+\mathcal{P})\frac{\dot\pi}{\pi} &= (1+\mathcal{P})\frac{Q}{C_vT} && (2.57)\\
\frac{\partial\pi_s}{\partial t} + \int_0^1 \nabla\,(m\mathbf{V})d\eta &= 0 && (2.58)
\end{aligned}
$$

2.6.2 Dynamical model diagnostic relations

Some diagnostic relations are used in order to compute various terms involved in the previous set of prognostic equations:

$$
\begin{aligned}
m &= (\partial\pi/\partial\eta) && (2.59)\\
p &= \pi(1+\mathcal{P}) && (2.60)\\
\phi &= \phi_s + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})}d\eta && (2.61)\\
D_3 &= \nabla\mathbf{V} + \frac{1}{m}\frac{p}{RT}\nabla\phi.\left(\frac{\partial\mathbf{V}}{\partial\eta}\right) + \mathsf{d} && (2.62)\\
m\dot\eta &= B\int_0^1 \nabla.m\mathbf{V}d\eta - \int_0^\eta \nabla.m\mathbf{V}d\eta && (2.63)\\
\dot\pi &= \left(\mathbf{V}.\nabla\pi - \int_0^\eta \nabla.m\mathbf{V}d\eta\right) && (2.64)\\
g\nabla w &= g\nabla w_s + \int_\eta^1 \frac{mRT}{p}\nabla\mathsf{d}\,d\eta' - \int_\eta^1 \mathsf{d}\,\frac{\partial\nabla\phi}{\partial\eta}d\eta' && (2.65)
\end{aligned}
$$

2.7 Final Form of Dynamical Model Equations (variables $\mathcal{P}$, $\mathbb{d}$)

2.7.1 Prognostic equations

$$\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla\phi = \mathcal{V} \tag{2.66}$$

$$\frac{d\mathbb{d}}{dt} + g^2\frac{p}{mRT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial\pi\mathcal{P}}{\partial\eta}\right) - g\frac{p}{mRT}\frac{\partial\mathbf{V}}{\partial\eta}.\nabla w - \mathbb{d}(\mathsf{X}-\mathbb{d}) - \dot{\mathsf{X}} = -g\frac{p}{mRT}\frac{\partial\mathcal{W}}{\partial\eta} \tag{2.67}$$

$$\frac{dT}{dt} + \frac{RT}{C_v}D_3 = \frac{Q}{C_v} \tag{2.68}$$

$$\frac{d\hat{q}}{dt} + \frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi} = \frac{Q}{C_vT} \tag{2.69}$$

$$\frac{\partial\pi_s}{\partial t} + \int_0^1 \nabla(m\mathbf{V})d\eta = 0 \tag{2.70}$$

2.7.2 Dynamical model diagnostic relations

Some diagnostic relations are used in order to compute various terms involved in the previous set of prognostic equations:

$$m = (\partial\pi/\partial\eta) \tag{2.71}$$

$$p = \pi\left[\exp(\hat{q}) - 1\right] \tag{2.72}$$

$$\phi = \phi_s + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})}d\eta \tag{2.73}$$

$$D_3 = \nabla\mathbf{V} + \mathbb{d} \tag{2.74}$$

$$m\dot{\eta} = B\int_0^1 \nabla.m\mathbf{V}d\eta - \int_0^\eta \nabla.m\mathbf{V}d\eta \tag{2.75}$$

$$\dot{\pi} = \left(\mathbf{V}.\nabla\pi - \int_0^\eta \nabla.m\mathbf{V}d\eta\right) \tag{2.76}$$

$$g\nabla w = g\nabla w_s + \int_\eta^1 \frac{mRT}{p}\nabla(\mathbb{d}-\mathsf{X})\,d\eta' - \int_\eta^1 (\mathbb{d}-\mathsf{X})\,\frac{\partial\nabla\phi}{\partial\eta}d\eta' \tag{2.77}$$

$$\mathsf{X} = \frac{p}{mRT}\nabla\phi\,.\left(\frac{\partial\mathbf{V}}{\partial\eta}\right) \tag{2.78}$$

Chapter 3

Associated Linear Continuous System

(18/09/2003)

3.1 Introduction

If we consider an atmosphere in a given state (noted symbolically $\mathcal{X}^*$), the instantaneous evolution of small amplitude disturbances around this state $\mathcal{X}^*$ can be deduced from the analysis of the linearised set of equations around $\mathcal{X}^*$ (noted symbolically $\mathcal{L}$). If $\mathcal{X}^*$ is stationary, the linearized system $\mathcal{L}$ allows to predict the long-term evolution of such small disturbances around this stationary-state $\mathcal{X}^*$ through:

$$\frac{\partial \mathcal{X}}{\partial t} = \mathcal{L}.\mathcal{X} \tag{3.1}$$

where $\mathcal{X}$ is a symbolic notation for the disturbance structure. The disturbances which have a periodic evolution in time can then be found as solution of a (complex) eigenmode problem, since they satisfy:

$$\mathcal{L}.\mathcal{X}_\omega = i\omega\mathcal{X}_\omega \tag{3.2}$$

where ω is the frequency of the mode and $\mathcal{X}_\omega$ the associated structure (eigenfunction). These periodic disturbances are called normal modes of the system around $\mathcal{X}^*$, provided they have a bounded energy density (cf: Bénard, 2003 and Bénard et al., 2004b). The analytical solution of this eigenproblem is not always possible. In fact the analysis of the normal modes is analytically possible only for some very simple $\mathcal{X}^*$ stationary-states. However, the analytic description of the normal modes for these very simple states is important since it allows to describe how fast disturbances (waves) propagate in the domain. This points out the theoretical importance of deriving the linearized system, at least for very simple states $\mathcal{X}^*$.

Moreover, there is another reason why the derivation of the linear system is important. It will be seen in Chapter 5 that the most efficient schemes are those who allow an implicit treatment of a part of the evolution terms of the complete system. However, the solution of the resulting implicit problem requires some linearization of the system (as already mentionned in section 1.1.4), because non-linear implicit systems cannot be solved by direct methods. This difficulty can be circumvented by using a linearized system $\mathcal{L}$, which then acts as a pre-conditioner for the solution of the implicit problem through a preconditioned generalized fixed-point iterative algorithm (see Bénard, 2003 for more details). This points out the practical importance of deriving the linearized system, for allowing an implicit treatment of the evolution terms.

Hence, the derivation of the linear system associated to the complete set of equation is important for both theoretical and prectical reasons.

3.2 Choice of the Linear System for Implicit Treatments

As stated in Bénard (2004), the choice of the linear system $\mathcal{L}$, used as a pre-conditioner to solve the implicit problem resulting from an implicit time treatment, is arbitrary. However, an "irrelevant" choice for $\mathcal{L}$ will not help to enhance the convergence towards the non-linear implicit solution and thus will not practically ensure the stability and robustness of the system. Hence a relevant choice of the $\mathcal{L}$ linear system used in the implicit treatment is important for allowing an efficient time discretisation.

In a general way, an optimal convergence/robustness could be expected when $\mathcal{L}$ is the tangent-linear system around the actual state of the atmosphere at the time t where the system is to be solved. This is because the non-linear residuals are then minimum. However, the implicit problem to be solved for such a linear system would be very complex. An attempt of this approach is found in Skamarock et al. (1997) , and in a lesser extent in Thomas et al. (1998). The problem is made complicated to solve because the coefficients of the linear system are then variable in time and in the horizontal directions. Hence the solution of a complete 3D spatial partial differential equations system is needed at each time-step. This approach is better adapted for models which do not use the spectral transform method.

For spectral models like Aladin-NH, it is strategically convenient that the spectral transform plays the role of the horizontal part of the implicit solver. This condition is fulfilled if the coefficients of the linear system $\mathcal{L}$ are horizontally homogeneous. Moreover, if the coefficients of $\mathcal{L}$ are constant in time, the implicit system can be inverted at the level of the model set-up, thus resulting in a further efficiency. As a consequence, the choice of a linear system in which the coefficients are constant in time and horizontally, is a strategic choice for Aladin-NH. This approach is referred to as "constant-coefficients approach".

In the following series of papers:
- Bénard, 2003
- Bénard et al., 2004a
- Bénard, 2004
- Bénard et al., 2004b

it is shown that in spite of its simplicity, this constant-coefficients approach, which has extensively proven to be strategically relevant for the HPE system, is also strategically relevant for the EE system.

3.3 Derivation of the linear system

3.3.1 Basic state

Both for the theoretical analysis and in the semi-implicit and iterative-centred-implicit schemes of Aladin-NH, the linear operator $\mathcal{L}$ is based on the linearized set of equations around a very simple atmospheric state $\mathcal{X}^*$ which is:
- Resting
- Hydrostatically balanced
- isothermal
- horizontally homogeneous
- with a vanishing surface geopotential.

Following a classical notation, the basic state variables are represented by an asterisc.
We thus have:

$$
\begin{aligned}
T^* &= \mathrm{Cst} \\
u^* &= v^* = 0 \\
w^* &= \mathsf{d}^* = \mathbb{d}^* = 0 \\
\phi_s^* &= \phi^*_{(\eta=1)} = 0 \\
\mathcal{P}^* &= \hat{q} = 0 \\
\pi_s^* &= \mathrm{Cst}
\end{aligned}
$$

The basic state is horizontally homogeneous (along iso-η surfaces). The variables ϕ^*, π^* and m^* are functions of η only. The equation of state for the basic state is:

$$\frac{d\phi^*}{d\eta} = -RT^* \frac{m^*}{\pi^*} \tag{3.3}$$

The vertical integration of the previous equation shows that the function $(\phi^* + RT^* \ln \pi^*)$ is independant of η:

$$\phi^* + RT^* \ln \pi^* = \mathrm{Cst}$$

We set the value of this constant by specifying the hydrostatic pressure at ground in the basic state:

$$\pi^*_{(\eta=1)} = \pi^*_s (= \mathrm{Cst})$$

The true pressure in the basic state p^* is of course equal to its hydrostatic counterpart π^* since the basic state is under hydrostatic equilibrium: $p^* = \pi^*$, thus we have $\rho^* = (\pi^*/RT^*)$.
The basic state is now entirely defined. One can see that it is described by the choice of two arbitrary constants: T^*, and π^*_s. This choice has some implications in terms of stability of the semi-implicit scheme (see Simmons *et al.* (1978), and the 2003-2004 series of papers by Bénard *et al.* on the SI stability).
The vertical profile of pressure in the basic state is a consequence of the choice of the A and B functions defining the vertical coordinate:

$$\pi^*_{(\eta)} = A_{(\eta)} + B_{(\eta)} \pi^*_s \tag{3.4}$$

$$m^*_{(\eta)} = \frac{dA}{d\eta} + \frac{dB}{d\eta} \pi^*_s \tag{3.5}$$

3.3.2 Definition of the deviation

For a given state of the atmosphere in the complete system, we define the deviation by the difference with the local value in the basic state. Classically, the quantities relatives to the deviation are primed. For the temperature and dynamical variables, we have thus:

$$
\begin{aligned}
T &= T^* + T' \\
\phi &= \phi^*_{(\eta)} + \phi' \\
\phi_s &= \phi_s^* = 0 \\
u &= u^* + u' = u' \\
v &= v^* + v' = v' \\
w &= w^* + w' = w' \\
D &= D^* + D' = D'
\end{aligned}
$$

where $D = (\partial u/\partial x) + (\partial v/\partial y)$ represents the horizontal part of the wind divergence. Since $u^* = v^* = w^* = D^* = 0$, we use the original variables u, v, D without primes in the linear model formulation.
If π and π_s represent the local values of hydrostatic pressure in the system, the deviation for current level and ground hydrostatic pressure are given by:

$$\begin{aligned}
\pi &= \pi^*_{(\eta)} + \pi' \\
\pi_s &= \pi^*_s + \pi'_s \\
m = &= m^* + m'
\end{aligned}$$

The variable π and its derivatives can be expressed in term of the ground values and A, B functions:

$$\begin{aligned}
\pi^* &= A(\eta) + B(\eta)\pi^*_s \\
\pi' &= B\pi'_s \\
m' &= \frac{dB}{d\eta}\pi'_s
\end{aligned}$$

Additionally we have:

$$\mathcal{P}' = \mathcal{P} \tag{3.6}$$
$$\hat{q}' = \hat{q} \tag{3.7}$$
$$\mathbb{d}' = \mathbb{d} \tag{3.8}$$
$$\mathsf{d}' = \mathsf{d} \tag{3.9}$$

Important remark: It is important to note that since $\mathbb{d} = \mathsf{d} + \mathsf{X}$ and X is a non-linear term, then, in the linear model, $\mathbb{d} = \mathsf{d}$. Similarly, since $(\hat{q} - \mathcal{P}) = \ln(1+\mathcal{P}) - \mathcal{P}$ is a non-linear term, we have in the linear model $\hat{q} = \mathcal{P}$. As a consequence, the form of the linear system is independent of the choice of the prognostic variables d or $\mathbb{d}$ on one hand, and $\hat{q}$ or $\mathcal{P}$ on the other hand. The following derivations are arbitrarily presented using the set of variables $(\mathcal{P}, \mathsf{d})$, but are formally valid for another combination of these nonhydrostatic prognostic variables.

Since $p = \pi + \pi\mathcal{P}$, the exact derivatives of p are:

$$\begin{aligned}
\nabla p &= (1+\mathcal{P})B\nabla\pi_s + \pi\nabla\mathcal{P} \\
\frac{\partial p}{\partial \eta} &= (1+\mathcal{P})m + \pi\frac{\partial \mathcal{P}}{\partial \eta}
\end{aligned}$$

3.3.3 Definition of the linearised system

Surface pressure tendency :

The linearisation of the the surface pressure tendency equation (2.28) gives:

$$\frac{\partial \pi'_s}{\partial t} + \nabla . \int_0^1 m^* \mathbf{V} d\eta = 0$$

that is:

$$\frac{\partial \pi'_s}{\partial t} + \int_0^1 m^* D d\eta = 0$$

Vertical divergence equation :

In the equation of the vertical divergence (2.55), the only source term giving a linear contribution is the first one, thus:

$$\frac{\partial \mathsf{d}}{\partial t} + g^2 \frac{\rho^*}{m^*} \frac{\partial}{\partial \eta} \left(\frac{1}{m^*} \frac{\partial \pi^* \mathcal{P}}{\partial \eta} \right) = 0$$

Thermodynamic equation :

The thermodynamic equation linearises to:

$$\frac{\partial T'}{\partial t} = -\frac{RT^*}{C_v} (D + \mathsf{d})$$

since the linear form of D_3 is:

$$D_3 \rightarrow (D + \mathsf{d})$$

Omega equation

The linearised version of $\dot{\pi}$ (also noted traditionnally ω) is:

$$\omega' = B \frac{\partial \pi_s}{\partial t} + m^* \dot{\eta}$$

and the linearised version of the omega equation is:

$$\omega' = -\int_0^\eta m^* D d\eta$$

Pressure deviation equation :

The linearisation of the $\mathcal{P}$ equation writes:

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{C_p}{C_v} (D + \mathsf{d}) + \frac{1}{\pi^*} \int_0^\eta m^* D d\eta$$

State equation :

As $\pi^*(d\phi^*/d\eta) = -RT^* m^*$, the state equation can be linearised to:

$$\pi^* \frac{\partial \phi'}{\partial \eta} + \pi' \frac{d\phi^*}{d\eta} + \pi^* \mathcal{P} \frac{d\phi^*}{d\eta} = -m' RT^* - m^* RT'$$

that is:

$$\frac{\partial \phi'}{\partial \eta} = m^* RT^* \frac{\pi'}{\pi^{*2}} + m^* RT^* \frac{\mathcal{P}}{\pi^*} - m' \frac{RT^*}{\pi^*} - \frac{m^*}{\pi^*} RT'$$

The vertical integration gives the expression of the geopotential deviation:

$$\int_\eta^1 \frac{\partial \phi'}{\partial \eta} d\eta = -RT^* \left[\frac{\pi'}{\pi^*} \right]_\eta^1 - R \int_\eta^1 \frac{m^*}{\pi^*} T' d\eta + RT^* \int_\eta^1 \frac{\mathcal{P}}{\pi^*} m^* d\eta$$

which takes the final form:

$$\phi' = -RT^* \frac{\pi'}{\pi^*} + RT^* \frac{\pi'_s}{\pi^*_s} + R \int_\eta^1 \frac{m^*}{\pi^*} T' d\eta - RT^* \int_\eta^1 \frac{\mathcal{P}}{\pi^*} m^* d\eta$$

Divergence equation :

The vector linearised momentum equation writes:

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{RT^*}{\pi^*}\left(\nabla \pi' + \pi^* \nabla \mathcal{P}\right) + \nabla \phi' = 0$$

taking the divergence of this equation leads to:

$$\frac{\partial D}{\partial t} = -\frac{RT^*}{\pi^*}\left(\Delta \pi' + \pi^* \Delta \mathcal{P}\right) - \Delta \phi'$$

and finally, using the above geopotential expression:

$$\frac{\partial D}{\partial t} = -R\int_\eta^1 \frac{m^*}{\pi^*}\Delta T' d\eta + RT^*\int_\eta^1 \frac{m^*}{\pi^*}\Delta \mathcal{P} d\eta - RT^*\Delta\mathcal{P} - \frac{RT^*}{\pi_s^*}\Delta \pi_s' \tag{3.10}$$

3.3.4 Notations

Some notations are introduced here for conciseness. We define here the basic-state square of the acoustic phase speed, the characteristic height of the atmosphere, and the square of Brünt-Vaisalä frequency for the reference-state:

$$\begin{aligned}
c_*^2 &= RT^*\frac{C_p}{C_v} \\
H_* &= \frac{RT^*}{g} \\
N_*^2 &= \frac{g^2}{C_p T^*}
\end{aligned}$$

One also defines a set of vertical continuous linear operators by:

$$\begin{aligned}
\partial^* X &= \frac{\pi^*}{m^*}\frac{\partial X}{\partial \eta} \\
\mathcal{G}^* X &= \int_\eta^1 \frac{m^*}{\pi^*} X\, d\eta \\
\mathcal{S}^* X &= \frac{1}{\pi^*}\int_0^\eta m^* X\, d\eta \\
\mathcal{N}^* X &= \frac{1}{\pi_s^*}\int_0^1 m^* X\, d\eta \\
\mathcal{L}^* X &= \partial^*\left(\partial^* + 1\right) X
\end{aligned}$$

The following properties are true for these operators:

$$\partial^*.\mathcal{G}^*.X = -X \tag{3.11}$$

$$\mathcal{G}^*.\partial^*.X = X_{(\eta=1)} - X \tag{3.12}$$

$$\left(\partial^* + 1\right).\mathcal{S}^*.X = \mathcal{S}^*.\left(\partial^* + 1\right).X = X \tag{3.13}$$

The evolution equations for perturbation in the linear system then writes:

$$\frac{\partial D}{\partial t} = -R\mathcal{G}^*\Delta T' + gH_*\mathcal{G}^*\Delta\mathcal{P} - RT^*\Delta\mathcal{P} - \frac{RT^*}{\pi_s^*}\Delta\pi_s' - \Delta\phi_s' \quad (3.14)$$

$$\frac{\partial T'}{\partial t} = -\frac{RT^*}{C_v}(D + \mathsf{d}) \quad (3.15)$$

$$\frac{\partial \pi_s'}{\partial t} = -\pi_s^*\mathcal{N}^* D \quad (3.16)$$

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{S}^* D - \frac{C_p}{C_v}(D + \mathsf{d}) \quad (3.17)$$

$$\frac{\partial \mathsf{d}}{\partial t} = -\frac{g^2}{RT_*}\mathcal{L}^*\mathcal{P} \quad (3.18)$$

3.3.5 Modification of the linearized vertical momentum equation

As pointed in section 1.1, the $\mathcal{L}$ linear system used for the linear separation of the implicit treatment is arbitrary. The most commonly used approach to determine $\mathcal{L}$ is to define and seek it as the tangent-linear operator around a given (and still arbitrary) reference-state $\mathcal{X}^*$. However, this approach introduces a restriction in the set of the operators which can be obtained for the separation. It was shown in Bénard (2004) that this unnecessary limitation could restrict the robustness that can be expected from implicit schemes based on the linear-separation method.

More specifically, it was shown in this latter paper that the robustness of the implicit treatments in Aladin-NH can be substantially modified if a specific reference-temperature value T_e^* is chosen in the terms involving the vertical propagation of elastic waves, i.e. the RHS term of the linearized vertical momentum equation (3.18). The robustness is increased if:

$$T_e^* < T^* \quad (3.19)$$

In the following, we note:

$$T_e^* = rT^* \quad (r \leq 1) \quad (3.20)$$

Important: The specification of a cold T_e^* is absolutely required for using Aladin-NH with 2-TL schemes, as shown in Bénard (2004).

The linear system (3.14) – (3.18) for the implicit problem then writes:

$$\frac{\partial D}{\partial t} = -R\mathcal{G}^*\Delta T' + gH_*\mathcal{G}^*\Delta\mathcal{P} - RT^*\Delta\mathcal{P} - \frac{RT^*}{\pi_s^*}\Delta\pi_s' \quad (3.21)$$

$$\frac{\partial T'}{\partial t} = -\frac{RT^*}{C_v}(D + \mathsf{d})$$

$$\frac{\partial \pi_s'}{\partial t} = -\pi_s^*\mathcal{N}^* D$$

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{S}^* D - \frac{C_p}{C_v}(D + \mathsf{d})$$

$$\frac{\partial \mathsf{d}}{\partial t} = -\frac{g}{rH_*}\mathcal{L}^*\mathcal{P} \quad (3.22)$$

3.3.6 Linearization of the map factor

At this stage, it is important to point out how the transformations occur in ARPEGE/Aladin. Spectral computations use the horizontal derivatives on the map ∇' and the wind images $\mathbf{V}' = (u', v')$. In fact, only the divergence on the map $D' = \nabla'.\mathbf{V}'$ and the vorticity on the map $\zeta' = \nabla' \times \mathbf{V}'$ are used in spectral computations. On the other hand, grid point computations apply to the components of the physical wind $\mathbf{V} = (u, v)$ and to the physical horizontal derivative ∇. As a consequence, a special process takes place during the integration time-loop, to switch alternatively from the (ζ', D') system to (u, v) one and *vice versa*.
To summarise schematically, let say that at the beginning of the time step, (ζ', D') are available in spectral coefficients. During subroutine ELINV, (u', v') are computed and stored, so that at the beginning of subroutine CPG, the two kinds of dynamical variables are available: (ζ', D', u', v'). In subroutine CPG, the variables are first transformed to geographical variables (ζ, D, u, v), the explicit guess at $t + \Delta t$ and the explicit part of the semi-implicit correction are computed, but only for (u, v). At the end of subroutine CPG, these terms are transformed back to reduced variables, so CPG is left with: $(u_E'^{+}, \delta u_{\mathsf{Lin}}'^{+}, v_E'^{+}, \delta v_{\mathsf{Lin}}'^{+})$ in grid point values. In subroutine CPGLAG the wo terms for each variable are summed giving the quantities to be passed in input for solving the implicit system: $(\widetilde{u'^{+}}, \widetilde{v'^{+}})$. In subroutine ELDIR, these terms are computed for the (ζ', D') system and stored, so that subroutine ESPC begins with $(\widetilde{u'^{+}}, \widetilde{u'^{+}}, \widetilde{\zeta'^{+}}, \widetilde{D'^{+}})$ available. In subroutine ESPC the implicit system is solved for (ζ', D'), providing these variables in spectral coefficients at time $t + \Delta t$.
(For further details, see ARPEGE note n° 22).
Since spectral computations are made with map variables and derivatives in the ARPEGE/Aladin grid coordinate system, the map factor has to be taken into account in the linear system. If the map factor is denoted $\mathbf{m}$, the horizontal pseudo divergence, gradient and laplacian operators write:

$$D' = D/\mathbf{m}^2 \tag{3.23}$$

$$\nabla' = \nabla/\mathbf{m} \tag{3.24}$$

$$\Delta' = \Delta/\mathbf{m}^2 \tag{3.25}$$

Finally, this system is linearised with respect to $\mathbf{m}$ in order to get horizontally constant coefficients for the variables. The map factor $\mathbf{m}$ is thus replaced by its maximum value $\mathbf{m}_*$:

$$\mathbf{m}_* = \max_{(\text{domain})} [\mathbf{m}] \tag{3.26}$$

This simplification appears to be legitimate for limited area models in which $\mathbf{m}$ remains close to unity.
In the global stretched ARPEGE model, where the map factor reaches large values, this simplification cannot be applied. In this case $\mathbf{m}$ is linearized into its actual value (i.e. $\mathbf{m}_* = \mathbf{m}$) and the solution of the implicit system results in the inversion of penta-diagonal matrices in the spectral space because the map factor is a function of the two first total wave-numbers (see Yessad and Bénard, 1996). This strategy is controlled by the namelist switch LSIDG in the ARPEGE code. Similarly, in view of limited-area applications with large domains in Mercator geometry, a provision is made to possibly use a linearization of the map factor around a non-constant value, close from the actual map-factor and function of the two first meridional wave-numbers. In this case, the multiplication by the linearized map factor becomes a spatially-variable spatial operator which does not commute with the spatial operators ∇' and Δ'. The following derivations take into account this non-commutativity in view of a LSIDG-type strategy in Aladin, and therefore the linearized map-factor $\mathbf{m}_*$ is assumed as non-commutative with respect to ∇' and Δ'. The mutual position of these operators is therefore fully relevant and important.

The linear system (3.21) – (3.22) for the implicit problem then writes:

$$\begin{aligned}
\frac{\partial D'}{\partial t} &= -R\mathcal{G}^*\Delta' T' + gH_*\mathcal{G}^*\Delta'\mathcal{P} - RT^*\Delta'\mathcal{P} - \frac{RT^*}{\pi_s^*}\Delta'\pi_s' && (3.27)\\
\frac{\partial T'}{\partial t} &= -\frac{RT^*}{C_v}(\mathbf{m}_*^2 D' + \mathsf{d}\,) && (3.28)\\
\frac{\partial \pi_s'}{\partial t} &= -\pi_s^*\mathcal{N}^*\mathbf{m}_*^2 D' && (3.29)\\
\frac{\partial \mathcal{P}}{\partial t} &= \mathcal{S}^*\mathbf{m}_*^2 D' - \frac{C_p}{C_v}(\mathbf{m}_*^2 D' + \mathsf{d}\,) && (3.30)\\
\frac{\partial \mathsf{d}}{\partial t} &= -\frac{g}{rH_*}\mathcal{L}^*\mathcal{P} && (3.31)
\end{aligned}$$

where d still represents indifferently d or $\mathbb{d}$. This latter system is the final form of the linear system actually used in Aladin-NH.

3.4 Structure Equation

3.4.1 Derivation of the Structure Equation

The structure equation is obtained by eliminating all variables but one in the previous linear system. The time-derivative of the d equation yields:

$$\begin{aligned}
\frac{\partial^2 \mathsf{d}}{\partial t^2} &= -\frac{g}{rH_*}\mathcal{L}^*\left[\mathcal{S}^*\mathbf{m}_*^2 D' - \frac{C_p}{C_v}(\mathbf{m}_*^2 D' + \mathsf{d}\,)\right]\\
\Rightarrow \quad \frac{\partial^2 \mathsf{d}}{\partial t^2} &= -\frac{g}{rH_*}\mathcal{L}^*\mathcal{S}^*\mathbf{m}_*^2 D' + \frac{c_*^2}{rH_*^2}\mathcal{L}^*\left(\mathbf{m}_*^2 D' + \mathsf{d}\right)
\end{aligned}$$

which can be written:

$$\left(\frac{\partial^2}{\partial t^2} - c_*^2\frac{\mathcal{L}^*}{rH_*^2}\right)\mathsf{d} = \frac{\mathcal{L}^*}{rH_*^2}\left(-gH_*\mathcal{S}^* + c_*^2\right)\mathbf{m}_*^2 D' \qquad (3.32)$$

The time derivative of the divergence equation yields:

$$\frac{\partial^2 D'}{\partial t^2} = -R\mathcal{G}^*\Delta'\frac{\partial T'}{\partial t} + gH_*\mathcal{G}^*\Delta'\frac{\partial \mathcal{P}}{\partial t} - RT^*\Delta'\frac{\partial \mathcal{P}}{\partial t} - \frac{RT^*}{\pi_s^*}\Delta'\frac{\partial \pi_s'}{\partial t}$$

i.e:

$$\frac{\partial^2 D'}{\partial t^2} = c_*^2\Delta'(\mathbf{m}_*^2 D' + \mathsf{d}\,) - gH_*\mathcal{G}^*\Delta'\mathsf{d} - gH_*\mathcal{A}_1^*\Delta'\mathbf{m}_*^2 D'$$

with:

$$\mathcal{A}_1^* = (-\mathcal{G}^*\mathcal{S}^* + \mathcal{G}^* + \mathcal{S}^* - \mathcal{N}^*)$$

It can easily be checked that this $\mathcal{A}_1^*$ operator is zero in this continuous approach. This constraint will also have to be fulfilled for the discrete operators case.
The divergence equation then become:

$$\left(\frac{\partial^2}{\partial t^2} - c_*^2\Delta'\mathbf{m}_*^2\right)D' = \left(-gH_*\mathcal{G}^* + c_*^2\right)\Delta'\mathsf{d} \qquad (3.33)$$

We note that:

$$\left(\frac{\partial^2}{\partial t^2} - c_*^2\mathbf{m}_*^2\Delta'\right).\left[\frac{\mathcal{L}^*}{rH_*^2}\left(-gH_*\mathcal{S}^* + c_*^2\right)\mathbf{m}_*^2\right]D' = \left[\frac{\mathcal{L}^*}{rH_*^2}\left(-gH_*\mathcal{S}^* + c_*^2\right)\mathbf{m}_*^2\right].\left(\frac{\partial^2}{\partial t^2} - c_*^2\Delta'\mathbf{m}_*^2\right)D'$$

Substituting (3.32) in the LHS and (3.33) in the RHS of the latter equation yields:

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - c_*^2 \mathbf{m}_*^2 \Delta'\right)\left(\frac{\partial^2}{\partial t^2} - c_*^2 \frac{\mathcal{L}^*}{rH_*^2}\right)\mathsf{d} &= \left[\frac{\mathcal{L}^*}{rH_*^2}\left(-gH_*\mathcal{S}^* + c_*^2\right)\mathbf{m}_*^2\right]\left(-gH_*\mathcal{G}^* + c_*^2\right)\Delta'\mathsf{d} \\ &= \frac{\mathcal{L}^*}{rH_*^2}\left(-gH_*\mathcal{S}^* + c_*^2\right)\left(-gH_*\mathcal{G}^* + c_*^2\right)\mathbf{m}_*^2\Delta'\mathsf{d}\end{aligned}$$

The previous algebraic manipulation did not require vertical operators to mutually commute. The same remark apply for horizontal operators.
The latter equation develops in:

$$\frac{\partial^4}{\partial t^4} - c_*^2 \frac{\partial^2}{\partial t^2}\left(\frac{\mathcal{L}^*}{rH_*^2} + \mathbf{m}_*^2\Delta'\right)\mathsf{d} - g^2\mathcal{L}^*\left(\mathcal{S}^*\mathcal{G}^* - \frac{C_p}{C_v}\mathcal{S}^* - \frac{C_p}{C_v}\mathcal{S}^*\right)\frac{1}{r}\mathbf{m}_*^2\Delta'\mathsf{d} = 0$$

An operator $g^2\mathcal{L}^*.\mathcal{A}_2^*$ applying to $(1/r)\mathbf{m}_*^2\Delta'\mathsf{d}$ appears. It is easy to show that this operator simplifies by virtue of the above mentionned properties of the continuous operators (3.11), (3.13):

$$g^2\mathcal{L}^*.\mathcal{A}_2^*.\mathbf{m}_*^2\Delta'\mathsf{d} = g^2\mathcal{L}^*\left(\mathcal{S}^*\mathcal{G}^* - \frac{C_p}{C_v}\mathcal{S}^* - \frac{C_p}{C_v}\mathcal{G}^*\right)\mathbf{m}_*^2\Delta'\mathsf{d} = N_*^2 c_*^2\mathbf{m}_*^2\Delta'\mathsf{d}$$

Here also, this property of continuous operators will have to be taken into account when designing the discrete operators. We finally obtain the structure equation of the system:

$$\left[-\frac{1}{c_*^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\mathbf{m}_*^2\Delta' + \frac{\mathcal{L}^*}{rH_*^2}\right) + \frac{N_*^2}{r}\mathbf{m}_*^2\Delta'\right]\mathsf{d} = 0 \tag{3.34}$$

This equation describes the structure (in the 4D space) of the perturbations allowed by the linear system. In particular, this structure equation determines the geometrical structure and propagative properties of linear waves, that is, those perturbations for which the evolution is periodic in time.

3.4.2 Comments

First, it is important to note that the obtained structure equation is formally identical to the one obtained for the same isothermal atmosphere for the fully compressible z system (cf: Eckart, 1960), simply replacing ∂^* by: $-H(\partial/\partial z)$ and r by 1. It is also identical to the one obtained for the fully compressible π system in the same conditions (cf: Laprise, 1992), simply replacing ∂^* by: $\pi(\partial/\partial\pi)$ and also r by 1. This simply reflects the fact that for a given atmosphere, the way waves propagate physically is independent of the chosen coordinate. As a consequence, the three systems behave exactly the same way for the waves they allow, and for their propagation characteristics. For example, all of them contain the possible propagation of the three rapid waves which are present in the real atmosphere: gravity waves, acoustic waves and Lamb waves.
Another important property of this equation is to be essentially a *local* one, only involving local derivative operators. Unlike in the hydrostatic system, (cf: ARPEGE documentation e.g.) no integral operator is to be considered for knowing the local propagation of waves.
Last, it should be noted that the coefficients of the operators are *constant* in time and along η surfaces. The consequence is that for a spectral model, the solution of the structure equation reverts to the inversion of a set of one constant coefficient Helmholtz equation per level. The solution of this type of equation doesn't present any particular problem, and is analogous to the horizontal Laplacian operator inversion in the hydrostatic case.

When $r \leq 1$ the propagation of the waves is modified. It can be shown that the geometrical structure of the waves is the same as for $r = 1$, and only the frequency of the waves is modified.

The stucture equation is valid for horizontally varying map-factors $\mathbf{m}_*$ as well as for uniform ones.

Note: We remark here that eliminating in favour of D instead of d would have needed the following rule:

$$\partial^*.\mathcal{G}^*.\mathcal{L}^* = \partial^*.\mathcal{L}^*.\mathcal{G}^*$$

and would have led to the following structure equation:

$$\partial^*.\left[-\frac{1}{c_*^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\mathbf{m}_*^2\Delta' + \frac{\mathcal{L}^*}{rH_*^2}\right) + \frac{N_*^2}{r}\mathbf{m}_*^2\Delta'\right] D = 0 \qquad (3.35)$$

These differences are of course of no importance in the continuous framework considered here, but it will become important in the discretised context.

Chapter 4

Vertical Discretisation

(18/09/2003)

In this chapter, the vertical discretisation of Aladin-NH is described. The horizontal discretisation is a spectral one, hence there are no specificity about this horizontal discretisation for the EE version compared to the hydrostatic version. The vertical discretisation of Aladin-NH is a finite-difference discretisation (a finite-element vertical discretisation exists for the ARPEGE model, but has not yet been adapted to Aladin).

Building a space-discretisation for a model could be viewed as an easy task, in which many arbitrary discretisation choices could be made without justification. However, this simplistic view is erroneous. Doing this way would most certainly lead to an unstable model, which additionnally would not respect basic physical rules valid for the continuous equations (conservation of invariants, local elimination of operators,...).

Hence the most rational way to perform the discretisation for a model is to take the problem in the reverse direction, i.e. to determine the discretisation in such a way that the as many as possible of the interesting properties of the continuous equations should be reflected by similar properties in the discretised model. The continuous properties are thus used as constraints for the design of the vertical discretisation.

However, it should be outlined that using this strategy for determining the discretisation choices results in a complicated problem with many simultaneous unknowns. However, among all the properties of the continous equations, some represent constraints which are much more important than others for the discretised model. Hence the problem of the determination of the discretisation can be simplified by trying to fulfil in priority the most important constraints. A hierarchy can be established as follows:

- Constraints which are simply mandatory for being able to write the discretised equations. An example of this can be found in the constraint (C1) used for the derivation of the discretized wave-structure Helmholtz equation (cf: section 4.2).
- Constaints which are necessary for guarding the discretised model against instability. An example of this can be found in the non-hydrostatic part of the angular-momentum conservation constraint (cf: section 4.11.1).
- Constraints which are necessary for insuring a conservation of a discretised counterpart of the continuous invariants of the original equations system. Examples of this can be found in the hydrostatic part of the angular-momentum conservation constraint (cf: section 4.11.1), and in the energy conservation constraint (cf: section 4.10).

Some people in NWP groups argue that for a short-range limited-area model, the constraints linked to the last category can be ignored for simplicity, if desired. Nevertheless, the Aladin-NH discretisation has been built in

such a way that these constraints are fulfilled. It will be seen in section 4.11.1 that the only exception is for the conservation of the angular-momentum for the non-hydrostatic deviation of the pressure force, because formally fulfilling this constraint would have made the model unstable, and thus would have violated a more important constraint.
The Aladin-NH discretisation scheme is thus energy conservaing and "hydrostatic" angular-momentum conserving.

4.1 Definitions

The vertical domain is discretised in L layers indexed from 1 at the top to L at the bottom. The generic index for these layers is noted l. Each layer is separated from his neighbours by interfaces, the generic index of which is $\tilde{l}$. The interfaces are indexed from $\widetilde{0}$ at the top boundary to $\widetilde{L}$ at the bottom boundary, as shown on the figure 4.1. The grid is vertically staggered such as all 3D prognostic variables are defined in the layers (dashed lines, levels of type l), and the vertical fluxes at the interfaces (solid lines, levels of type $\tilde{l}$). The vertical velocity is also defined at interfaces, so that the vertical derivatives d and 𝕕 which are the actual prognostic variables of the model, are defined in the layers. A variable X defined in the layer l is noted X_l, and at the interface $X_{\tilde{l}}$.

The fact that the vertical coordinate η is implicit results in a direct definition of the hydrostatic pressure π at the interfaces:

$$\pi_{\tilde{l}} = A_{\tilde{l}} + B_{\tilde{l}}.\pi_{\tilde{L}}$$

which is the discrete form of the coordinate definition. The hydrostatic-pressure depth of a layer l is:

$$\delta\pi_l = \pi_{\tilde{l}} - \pi_{\tilde{l}-1}$$

4.2 Discretisation of the Linear Model

The linear model must be discretised first because the determination of its discretised structure (Helmholtz) equation is a constraint of the strongest class in the above hierarchy. In the linear model used for the semi-implicit scheme, the interfaces and thicknesses are taken to their basic-state value:

$$\begin{aligned}
\pi^*_{\tilde{l}} &= A_{\tilde{l}} + B_{\tilde{l}}.\pi^*_{\tilde{L}} \\
\delta\pi^*_{l} &= \pi^*_{\tilde{l}} - \pi^*_{\tilde{l}-1}
\end{aligned}$$

The discretisation of the linear model is obtained by taking the discrete counterpart of the space-continuous linear system (3.14)-(3.18). For this we define the discrete operators corresponding to the continuous integral operators introduced in section 3.3.4:

$$\begin{aligned}
(\mathbf{G}^*.X)_l &= \left[\sum_{k=l+1}^{L} \delta^*_k X_k + \alpha^*_l X_l\right] \\
(\mathbf{S}^*.X)_l &= \left[\frac{1}{\pi^*_l}\sum_{k=1}^{l-1} \delta\pi^*_k X_k + \beta^*_l X_l\right] \\
(\mathbf{N}^*.X)_l &= \frac{1}{\pi^*_{\tilde{L}}}\sum_{k=1}^{L} \delta\pi^*_k X_k
\end{aligned}$$

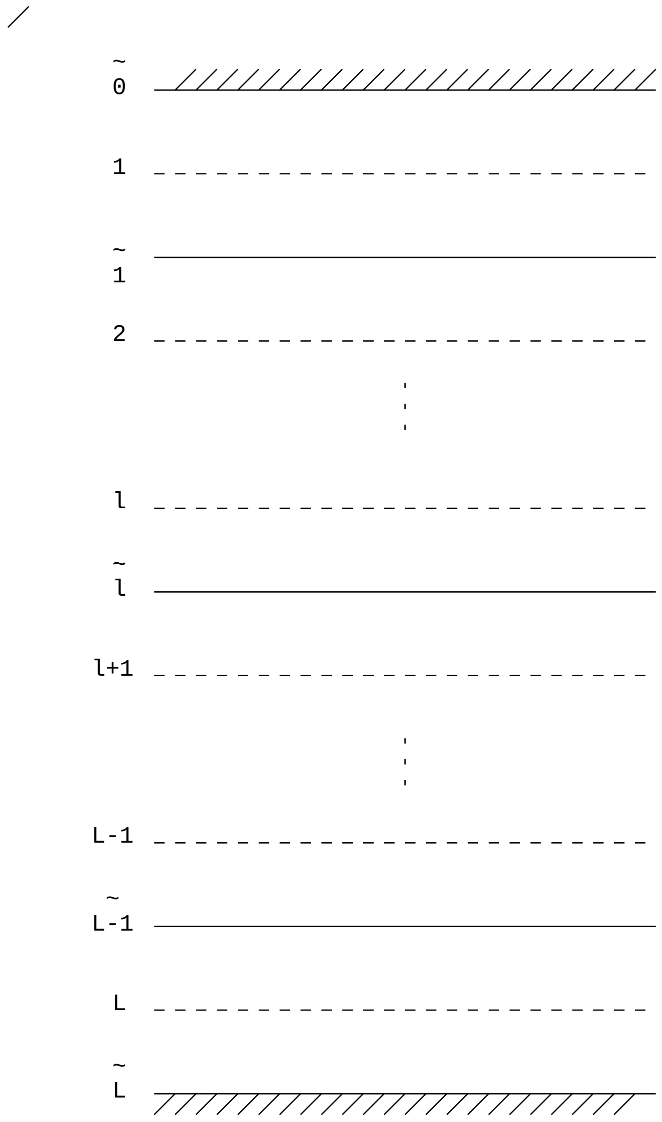

Figure 4.1: vertical staggering of Aladin-NH

where π^*_k is a discrete form of the basic-state hydrostatic-pressure in the layer k, and δ^*_k a discrete form for $d(\log\pi^*)$ in the layer k. T the quantities α^*_l and β^*_l are two dimensionless shifts for completing the discrete integrals from the closest interface level to the layer l level. The precise form of the four latter quantities $\pi^*_l, \delta^*_l, \alpha^*_l, \beta^*_l$ is, left undetermined for the moment, and will be chosen conveniently, to insure elimination, stability and conservation properties. The discrete counterpart of $\mathcal{L}^*$ is noted $\mathbf{L}^*$, and here also no special form is retained for it for time being.

The starting point is the linear system (3.14) – (3.18). The discretised system has the same shape in discrete form, provided that each continuous vertical operator (i.e: $\mathcal{L}^*$, $\mathcal{G}^*$,...) is replaced by its discrete equivalent (i.e: $\mathbf{L}^*$, $\mathbf{G}^*$,...). Then it can be seen that the algebraic elimination is formally identical in discrete and continuous case, and a similar operator $\mathbf{A_1^*}$ appears in the discrete equivalent of the divergence equation (3.32). This operator writes:

$$\mathbf{A_1^*} = -\mathbf{G}^*\mathbf{S}^* + \mathbf{G}^* + \mathbf{S}^* - \mathbf{N}^*$$

we thus define a first constaint (C1) for the discrete operators by imposing that, as in the continuous case:

$$\underline{\text{(C1) :}} \qquad \mathbf{A_1^*} = 0 \tag{4.1}$$

The elimination then can be pursued identically to the continuous one, until the derivation of the final structure equation. It should be noted that with the elimination order chosen here (in favour od d and not D), no commutativity property has been used to derive the structure equation. The derivation thus leads to another constraint (C2) concerning the operator applying to $(1/r)\mathbf{m}_*^2\Delta'$d:

$$\underline{\text{(C2) :}} \qquad g^2\mathbf{L}^*.\mathbf{A_2^*} = g^2\mathbf{L}^*\left(\mathbf{S}^*\mathbf{G}^* - \frac{C_p}{C_v}\mathbf{S}^* - \frac{C_p}{C_v}\mathbf{G}^*\right) = N_*^2c_*^2 \tag{4.2}$$

Finally, to respect the local character of the Euler equations system and of its structure equation in the discretised context, the second-order vertical derivative operator $\mathbf{L}^*$ is required to be as compact as possible (i.e. tridiagonal) for a vertical column. Hence:

$$\underline{\text{(C3) :}} \qquad \mathbf{L}^* \text{ is a tridiagonal operator} \tag{4.3}$$

The problem is now to define the unknowns α^*_l, β^*_l, π^*_l, δ^*_l and $\mathbf{L}^*$ in such a way that these three constraints are fullfilled.

4.3 Determination of Discrete Vertical Operators in the Linear Model

4.3.1 First constraint

In order to examine the first constaint, we write the matrix of the $\mathbf{A_1^*}$ operator. We respectively note $\text{sup}_{(i,j)}$, $\text{inf}_{(i,j)}$ and $\text{diag}_{(i,i)}$, the generic superior, inferior, and diagonal term of the matrix for the row i and the column j. The operator $\mathbf{A_1^*}$ then writes:

$$\begin{aligned}
\text{sup}_{(i,j)} &= \delta^*_j(1-\beta^*_j) - \delta\pi^*_j\left(\frac{1}{\pi^*_{\tilde{L}}} + \sum_{k=j+1}^{L}\frac{\delta^*_k}{\pi^*_k}\right) \\
\text{diag}_{(i,i)} &= \alpha^*_i + \beta^*_i - \alpha^*_i\beta^*_i - \delta\pi^*_i\left(\frac{1}{\pi^*_{\tilde{L}}} + \sum_{k=i+1}^{L}\frac{\delta^*_k}{\pi^*_k}\right)
\end{aligned}$$

$$\mathsf{inf}_{(i,j)} = \frac{\delta\pi_j^*}{\pi_i^*}(1-\alpha_i^*) - \delta\pi_j^*\left(\frac{1}{\pi_{\tilde{L}}^*} + \sum_{k=i+1}^{L}\frac{\delta_k^*}{\pi_k^*}\right)$$

where undefined symbols are zero. The solution of this system leads to a first set of constaint between the four unknowns:

$$\begin{aligned}
\alpha_l^* &= 1 - \frac{\pi_l^*}{\pi_{\tilde{l}}^*} \\
\beta_l^* &= 1 - \frac{\pi_{\tilde{l}-1}^*}{\pi_l^*} \\
\frac{\delta_l^*}{\pi_l^*} &= \frac{1}{\pi_{\tilde{l}-1}^*} - \frac{1}{\pi_{\tilde{l}}^*}
\end{aligned}$$

4.3.2 Second constraint

The second constraint consists in identifying $g^2\mathbf{L}^*.\mathbf{A_2^*}$ to a N^2c^2 homothecy. The weight of the three levels involved by a given layer l in the tridiagonal $\mathbf{L}^*$ operator will be noted A_l^*, B_l^* and C_l^*; thus we have:

$$(\mathbf{L}^*.X)_l = \mathsf{A}_l^*.X_{l-1} + \mathsf{B}_l^*.X_l + \mathsf{C}_l^*.X_{l+1}$$

We proceed as for the first constraint, writing the matrix form of $\mathbf{A_2^*}$.

$$\begin{aligned}
\mathsf{sup}_{(i,j)} &= -\frac{R}{C_v}\delta_j^* \\
\mathsf{diag}_{(i,i)} &= -\frac{R}{C_v}(\alpha_i^* + \beta_i^*) \\
\mathsf{diag}_{(1,1)} &= -\frac{R}{C_v}(\alpha_1^* + \beta_1^*) + 1 \\
\mathsf{inf}_{(i,j)} &= -\frac{R}{C_v}\frac{\delta\pi_j^*}{\pi_i^*}
\end{aligned}$$

It can be shown that provided the first constraint is fullfilled, the matrix $\mathbf{L}^*.\mathbf{A_2^*}$ *cannot* be identified to an homothecy, since it would require $\alpha_l^* + \beta_l^* = \delta_l^*$, i.e. $\pi_l^* = \pi_{\tilde{l}}^*$, which means no more corrective shift for evaluating the integral operators in the layers. This solution should not be retained since it acts as a non-centered vertical scheme.
One can further show that requiring this second matrix to be purely diagonal is even incompatible with any convenient non trivial choice of the four unknowns, independently of whatever or not the first constraint is fullfilled.
On the other hand, our main goal is to be able to derive a discretised structure equation which reamins local, hence it is sufficient for our purpose that the operator $\mathbf{T}^* = (g^2/N_*^2c_*^2)\mathbf{L}^*.\mathbf{A_2^*}$ should be a tridiagonal operator (instead of purely diagonal), provided that this operator behaves physically as a local vertical average operator applying to d values.
The final form of the second constraint then becomes:

$$\underline{(\mathrm{C2}'):} \qquad g^2\mathbf{L}^*.\mathbf{A_2^*} = g^2\mathbf{L}^*\left(\mathbf{S}^*\mathbf{G}^* - \frac{C_p}{C_v}\mathbf{S}^* - \frac{C_p}{C_v}\mathbf{G}^*\right) = N_*^2c_*^2\mathbf{T}^* \tag{4.4}$$

The resolution of the constraint is written by imposing the weight of this local averaging to be equal to 1. The physical way to impose this is to consider the vertical shift in vertical velocity induced by an individual d_l value and by its "averaged" counterpart $(\mathbf{T}.\mathsf{d}\,)_l$. This shift must be equal in the two cases, in order to conserve the advective properties of the vertical velocity field. Writing equality between these two shifts leads to:

$$\delta w'_l = -H_* \frac{m^*}{\pi^*} \mathsf{d}_l = -H_* \frac{m^*}{\pi^*} \left(\mathbf{T}^* \mathsf{d}\right)_l$$

which expresses the physical normalisation constraint applying on the averaging operator $\mathbf{T}^*$.
After some algebraic manipulation, one shows that all the above constraints are fullfilled for the following solution:

$$\begin{aligned}
\alpha_l^* &= \beta_l^* = 1 - \sqrt{\frac{\pi_{\tilde{l}-1}^*}{\pi_{\tilde{l}}^*}} \\
\pi_l^* &= \sqrt{\pi_{\tilde{l}-1}^* . \pi_{\tilde{l}}^*} = \frac{\delta \pi_l^*}{\delta_l^*} \\
\delta_l^* &= \frac{\delta \pi_l^*}{\pi_l^*} \\
\mathsf{A}_l^* &= \frac{\pi_{l-1}^*}{\delta_l^* \left(\pi_l^* - \pi_{l-1}^*\right)} \\
\mathsf{B}_l^* &= -\frac{1}{\delta_l^*} \left(\frac{\pi_l^*}{\pi_l^* - \pi_{l-1}^*} + \frac{\pi_l^*}{\pi_{l+1}^* - \pi_l^*} \right) \\
\mathsf{C}_l^* &= \frac{\pi_{l+1}^*}{\delta_l^* \left(\pi_{l+1}^* - \pi_l^*\right)}
\end{aligned}$$

The determination of the values at the top yields:

$$\begin{aligned}
\alpha_1^* &= \beta_1^* = 1 \\
\delta_1^* &= \alpha_1^* + \beta_1^* + \frac{C_v}{R} = 2 + \frac{C_v}{R} = 1 + \frac{C_p}{R} \\
\pi_1^* &= \frac{\delta \pi_1^*}{\delta_1^*} \\
\mathsf{A}_1^* &= 0 \\
\mathsf{C}_1^* &= \frac{\pi_2^*}{\delta_1^* \left(\pi_2^* - \pi_1^*\right)} \\
\mathsf{B}_1^* &= -\mathsf{C}_1^*
\end{aligned}$$

The bottom of the domain is not really a particular case provided that $\mathsf{C}_L^* = 0$ is taken. Then:

$$\begin{aligned}
\mathsf{A}_L^* &= \frac{\pi_{L-1}^*}{\delta_L^* \left(\pi_L^* - \pi_{L-1}^*\right)} \\
\mathsf{B}_L^* &= -\frac{\pi_L^*}{\delta_L^* \left(\pi_L^* - \pi_{L-1}^*\right)}
\end{aligned}$$

For $l \in [1, L-1]$, the general expression for $\mathbf{L}^*$ can be rewritten, after simplification:

$$\mathbf{L}^*.X_l = \mathsf{A}_l^* (X_{l-1} - X_l) + \mathsf{C}_l^* (X_{l+1} - X_l) \tag{4.5}$$

The tridiagonal $\mathbf{T}^*$ matrix elements write, for the row i:

(diag) $$\mathbf{T}_{(i,i)}^* = 1 - \frac{1}{\delta_i^*} \left(\frac{\pi_i^*}{\pi_{i+1}^* - \pi_i^*} + \frac{\pi_i^*}{\pi_i^* - \pi_{i-1}^*} \right) (\delta_i^* - 2\alpha_i^*)$$

(diagattop) $$\mathbf{T}_{(1,1)}^* = 1$$

$$\text{(inf)} \qquad \mathbf{T}^*_{(i,i-1)} = \frac{1}{\delta_i^*}\frac{\pi_{i-1}^*}{\pi_i^* - \pi_{i-1}^*}\left(\delta_{i-1}^* - 2\alpha_{i-1}^*\right)$$
$$\text{(sup)} \qquad \mathbf{T}^*_{(i,i+1)} = \frac{1}{\delta_i^*}\frac{\pi_{i+1}^*}{\pi_{i+1}^* - \pi_i^*}\left(\delta_{i+1}^* - 2\alpha_{i+1}^*\right)$$

which can finally be written:

$$\mathbf{T}^* = (\mathbf{I} + \mathbf{L}^*.\mathbf{Q}^*) \tag{4.6}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{Q}^*$ is the diagonal matrix the row i diagonal element of which is given by $\mathbf{Q}^*_{(i,i)} = (\delta_i^* - 2\alpha_i^*)$ for $i \in [2, L]$, and $\mathbf{Q}^*_{(1,1)} = 0$.

4.3.3 Structure equation

The structure equation is then for the vertically discretized case:

$$\left[-\frac{1}{c_*^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\mathbf{m}_*^2\Delta' + \frac{\mathbf{L}^*}{rH_*^2}\right) + \frac{N_*^2}{r}\mathbf{T}^*\mathbf{m}_*^2\Delta'\right]\mathsf{d} = 0 \tag{4.7}$$

We now clearly see the "local" nature of the structure equation which is characteristic of the relaxation of hydrostatic and anelastic "global" effects. In the spectral space, the solution of this equation results in a tridiagonal inversion with a constant coefficients matrix, given the time step and the horizontal total wavenumber. This means that, like in the hydrostatic case, the inverse of the semi-implicit operator can be pre-computed for each total wavenumber, and stored at the beginning of the model integration.

The major formal difference between the space-continuous (3.34) and space-discretized (4.7) structure equations is the appearance of the $\mathbf{T}^*$ operator. The impact of this difference is examined in Chapter 10. It is shown that :

[-] the second-order accuracy of the scheme is maintained.

no amplitude error is introduced (the scheme is not made spuriously unstable or damping).

- the only noticeable effect is a further decrease of the shortest gravity modes phase velocity (which is already decreased by the SI scheme).

Finally, the overall impact of the matrix $\mathbf{T}^*$ is found to be rather small, and the relevance of the vertical discretisation can be checked by evaluating the forecast on academic cases, for which the exact response is known. The finite-difference vertical-discretisation described above is thought to be of comparable accuracy as for similar state-of-the-art models.

4.4 Discretisation of the Non-Linear Model

The consistency of the discretisations of the linear and non-linear models is required to maintain the stability. Hence, the discretisation of the complete non-linear model follows the rules edicted for the linear model. In particular, each discretisation parameter in the linear model has its own counterpart in the complete model.

4.4.1 Vertical levels

The discrete vertical level characteristics are taken under the same form as in the linear model:

$$
\begin{aligned}
\pi_l &= \sqrt{\pi_{\tilde{l}-1}.\pi_{\tilde{l}}} = \frac{\delta\pi_l}{\delta_l} \\
\delta_l &= \frac{\delta\pi_l}{\pi_l}
\end{aligned}
$$

with the special boundary condition at top:

$$
\begin{aligned}
\delta_1 &= \alpha_1 + \beta_1 + \frac{C_v}{R} = 2 + \frac{C_v}{R} = 1 + \frac{C_p}{R} \\
\pi_1 &= \frac{\delta\pi_1}{\delta_1}
\end{aligned}
$$

Similarly, the definition of the discretization parameters α_l and β_l is taken from the linear form:

$$
\alpha_l = \beta_l = 1 - \sqrt{\frac{\pi_{\tilde{l}-1}}{\pi_{\tilde{l}}}}
$$

with the special boundary condition at top:

$$
\alpha_1 = \beta_1 = 1
$$

4.4.2 Vertical operators

For the vertical integral operators, the same principle is adopted than in the linear model to perform summation from a boundary to the level of the layer l, by addition of a corrective term α_l:

$$
\left(\int_\eta^1 \frac{m}{\pi} X d\eta\right)_l = \sum_{k=l+1}^{L} X_k \delta_k + \alpha_l X_l \tag{4.8}
$$

$$
\left(\frac{1}{\pi}\right)\left(\int_0^\eta m X d\eta\right)_l = \frac{1}{\pi}\sum_{k=0}^{l-1} X_k \delta\pi_k + \alpha_l X_l \tag{4.9}
$$

$$
\left(\int_0^1 m X d\eta\right)_l = \sum_{k=1}^{L} X_k \delta\pi_k \tag{4.10}
$$

The second-order derivative operator involved in Eq. (2.55) is the non-linear analogue of $\mathbf{L}^*$ and applies to $\mathcal{P}$:

$$
[\mathbf{L}\mathcal{P}]_l = \left[\pi \frac{\partial}{\partial \pi}\left(\frac{\partial \pi \mathcal{P}}{\partial \pi}\right)\right]_l = \mathsf{A}_l \mathcal{P}_{l-1} + \mathsf{B}_l \mathcal{P}_l + \mathsf{C}_l \mathcal{P}_{l+1} \tag{4.11}
$$

Here also, the definition for the coefficients of $\mathbf{L}$ is chosen to reflect the ones of the linear model:

$$
\begin{aligned}
\mathsf{A}_l &= \frac{\pi_{l-1}}{\delta_l(\pi_l - \pi_{l-1})} \\
\mathsf{B}_l &= -\frac{1}{\delta_l}\left(\frac{\pi_l}{\pi_l - \pi_{l-1}} + \frac{\pi_l}{\pi_{l+1} - \pi_l}\right) \\
\mathsf{C}_l &= \frac{\pi_{l+1}}{\delta_l(\pi_{l+1} - \pi_l)}
\end{aligned}
$$

This choice finally corresponds to the following expression:

$$
\begin{aligned}
[\mathbf{L}\mathcal{P}]_l &= \left[\pi\frac{\partial}{\partial\pi}\left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)\right]_l = \frac{1}{\delta_l}\left[\left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)_{\tilde{l}} - \left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)_{\tilde{l}-1}\right] && (4.12)\\
\left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)_{\tilde{l}} &= \frac{\pi_{l+1}\mathcal{P}_{l+1} - \pi_l\mathcal{P}_l}{\pi_{l+1} - \pi_l} && (4.13)
\end{aligned}
$$

As in (4.5), the general expression for $\mathbf{L}$ can be simplified to involve only A and C, for $l \in [1, L-1]$:

$$
\mathbf{L}.\mathcal{P}_l = \mathsf{A}_l(\mathcal{P}_{l-1} - \mathcal{P}_l) + \mathsf{C}_l(\mathcal{P}_{l+1} - \mathcal{P}_l) \tag{4.14}
$$

The top and bottom conditions for the operator $\mathbf{L}$ are directly drawn from the physical conditions applied at the top and bottom boaders of the domain: elastic at top, and rigid at the bottom. These points are developped in the two next sections.

4.5 Elastic Upper Boundary Condition for the non-linear model

In the current version of the model, the elastic top boundary condition is expressed through:

$$
p_T = 0 \tag{4.15}
$$

which rewrites (still in the current version of the model):

$$
(\pi\mathcal{P})_T = 0 \tag{4.16}
$$

This choice imposes the shape of the Laplacian operator applied to $\mathcal{P}$ at the top of the domain in the vertical momentum equation. Following (4.12)-(4.13), the $\mathbf{L}.\mathcal{P}$ term is expressed at top as:

$$
[\mathbf{L}\mathcal{P}]_1 = \frac{1}{\delta_1}\left[\left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)_{\tilde{1}} - \left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)_T\right] \tag{4.17}
$$

where :

$$
\left(\frac{\partial\pi\mathcal{P}}{\partial\pi}\right)_T = \frac{\pi_1\mathcal{P}_1 - (\pi\mathcal{P})_T}{\pi_1 - \pi_T} = \mathcal{P}_1 \tag{4.18}
$$

In the general formalism of (4.11), this therefore leads to the following values of A, B, and C, at top:

$$
\begin{aligned}
\mathsf{A}_1 &= 0\\
\mathsf{C}_1 &= \frac{\pi_2}{\delta_1(\pi_2 - \pi_1)}\\
\mathsf{B}_1 &= -\mathsf{C}_1
\end{aligned}
$$

It can be noted that the application of an elastic condition at top leads to an expression for $\mathbf{L}$ which is the direct counterpart of the expression obtained for $\mathbf{L}^*$ at top, in the above section 4.3.2. This feature is important for the stability of the model when employed with a Semi-Implicit scheme.

4.6 Free-slip boundary conditions for the horizontal wind

For the horizontal components of the wind, a free-slip boundary condition is assumed at the outest top and bottom half-layers of the domain:

$$\mathbf{V}_T = \mathbf{V}_{\tilde{0}} = \mathbf{V}_1 \tag{4.19}$$

$$\mathbf{V}_s = \mathbf{V}_{\tilde{L}} = \mathbf{V}_L \tag{4.20}$$

$$\tag{4.21}$$

These conditions are required in order to express various terms involved in the next sections 4.7–4.9.

4.7 Rigid Bottom Boundary Condition for the non-linear model

The rigid bottom boundary condition is also introduced in the model through a condition on the Laplacian operator involved in the vertical momentum equation. The involved terms are the second LHS terms in (2.55) and (2.67). It is thus seen that the problem is to specify the vertical Laplacian $\mathbf{L}$ applied to $\mathcal{P}$ at the bottom of the domain. It can be shown (see Chapter ???) that the specification of the bottom value of $\mathbf{L}(\mathcal{P})$ must be expressed consistently with the transport scheme of the model (Eulerian or semi-Lagrangian), otherwise spurious circulations occur in the stationnary response to orographically forced flows (known as "chimney effect"). Hence for the Eulerian version of Aladin-NH, the rigid bottom BC must be expressed in an Eulerian way, and in the SL version, the rigid bottom BC must be expressed in a Lagrangian way.

4.7.1 "Eulerian" Rigid Bottom BC

For the level $l = L$ we have:

$$[\mathbf{L}(\mathcal{P})]_L = \frac{1}{\delta_L}\left[-\left(\frac{\partial \pi\mathcal{P}}{\partial \pi}\right)_{\tilde{L}-1} + \left(\frac{\partial \pi\mathcal{P}}{\partial \pi}\right)_s\right]$$

We now must find an expression for the last term. First, it should be noted from Eq.(2.21) that:

$$\dot{w} = \mathcal{W} + g\frac{\partial\,(p-\pi)}{\partial \pi}$$

thus:

$$g\frac{\partial}{\partial \pi}\,(\pi\mathcal{P})_s = \dot{w}_s - \mathcal{W}_s$$

A parcel located at ground will indefinitely stay at ground due to the material condition $\dot{\eta}(1) = 0$. Therefore, for such a parcel the vertical acceleration $\dot{w}_s$ is only governed by the flow $\mathbf{V}_s$ along the rigid surface ϕ_s:

$$g w_s = \mathbf{V}_s.\nabla\phi_s \qquad \Rightarrow \qquad g\dot{w}_s = \frac{\partial \mathbf{V}_s}{\partial t}.\nabla\phi_s + \mathbf{V}_s.\nabla\left(\mathbf{V}_s.\nabla\phi_s\right)$$

or:

$$g\dot{w}_s = \dot{\mathbf{V}}_s.\nabla\phi_s + \left(u_s^2\frac{\partial^2\phi_s}{\partial x^2} + 2u_s v_s\frac{\partial^2\phi_s}{\partial x\partial y} + v_s^2\frac{\partial^2\phi_s}{\partial y^2}\right)$$

We note J_s the last bracketed Jacobian surface term. The final expression for $[\mathbf{L}(\mathcal{P})]_L$ is then:

$$[\mathbf{L}(\mathcal{P})]_L = \frac{1}{\delta_L}\left[-\left(\frac{\pi_L\mathcal{P}_L - \pi_{L-1}\mathcal{P}_{L-1}}{\pi_L - \pi_{L-1}}\right) + \frac{1}{g^2}\left(\dot{\mathbf{V}}_s.\nabla\phi_s + J_s\right) + \frac{1}{g}\mathcal{W}_s\right] \tag{4.22}$$

In practice, for the Eulerian version of the model, it is assumed that the boundary condition specification is internal to the dynamical kernel, and thus $\mathcal{W}$ is not considered. Consistently, $\dot{\mathbf{V}}_s$ is assumed to only contain Coriolis, pressure-gradient and curvature terms (but no physical contribution).

4.7.2 Lagrangian Rigid Bottom Boundary Condition

In case of a semi-Lagrangian transport scheme, the vertical acceleration at the surface can be expressed in a Lagrangian fashion, and we have directly:

$$[\mathbf{L}(\mathcal{P})]_L = \frac{1}{\delta_L}\left[-\left(\frac{\pi_L\mathcal{P}_L - \pi_{L-1}\mathcal{P}_{L-1}}{\pi_L - \pi_{L-1}}\right) + \frac{1}{g}\left(\dot{w}_s - \mathcal{W}_s\right)\right] \tag{4.23}$$

In the time discretized context, the details of the computations are developped in section 5.9.

4.8 Expression of w related terms

We give here the discretised expressions for gw and $g\nabla w$. The term $g\nabla w$ is needed for d evolution in Eq. (2.55) while gw is not used for evolution but its expression is given only for the needs of the post-processing. From Eq. (2.32), the continuous expression for gw and $g\nabla w$ are:

$$\begin{aligned} gw &= gw_s + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})}\mathsf{d}.d\eta \\ g\nabla w &= g\nabla w_s + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})}\nabla\mathsf{d}.d\eta \end{aligned}$$

with $gw_s = \mathbf{V}_s.\nabla\phi_s$ and $g\nabla w_s = \nabla(\mathbf{V}_s.\nabla\phi_s)$. Finally, using the free-slip condition for $\mathbf{V}$:

$$\begin{aligned} gw_{\tilde{l}} &= \mathbf{V}_L.\nabla\phi_s + \sum_{k=l+1}^{L}\frac{RT_k}{(1+\mathcal{P}_k)}\,\delta\mathsf{d}_k \\ g\nabla w_{\tilde{l}} &= \nabla\left(\mathbf{V}_L.\nabla\phi_s\right) + \sum_{k=l+1}^{L}\frac{RT_k}{(1+\mathcal{P}_k)}\,\delta(\nabla\mathsf{d})_k \end{aligned} \tag{4.24}$$

When $\mathbb{d}$ is used as a prognostic variable, d_k had to be replaced by $(\mathbb{d}_k - \mathsf{X}_k)$ in the latter expressions.

4.9 Expression of non-advective $\frac{\partial \mathbf{V}}{\partial \eta}$ terms

Examination of the original system shows that two non-advective terms involving $(\partial\mathbf{V}/\partial\eta)$ appear, namely in (2.55) and (2.62). This kind of term is problematic since $\delta\mathbf{V}$ is naturally available only at levels $\tilde{l}$.
The evolution term $[(gp/mRT)(\partial\mathbf{V}/\partial\eta).\nabla w]$ in Eq.(2.55) can be rewritten as $[(\delta\mathbf{V}.g(1+\mathcal{P})\nabla w)_l\,/\,(RT\delta_l)]$. It can then be obtained by linearly interpolating the horizontal wind between vertical levels in the layers:

$$\mathbf{V}_{\tilde{l}} = \epsilon_l\mathbf{V}_l + (1-\epsilon_l)\mathbf{V}_{l+1} \tag{4.25}$$

with ϵ_l being the weights for linear interpolator:

$$\epsilon_l = \frac{\delta_{l+1} - \alpha_{l+1}}{\delta_{l+1} - \alpha_{l+1} + \alpha_l} \tag{4.26}$$

The final form of the evolution term is:

$$g\frac{(1+\mathcal{P})_l}{RT_l\delta_l}\left(\delta\mathbf{V}.\nabla w\right)_l = \frac{g}{RT_l\delta_l}\left[\left(\mathbf{V}_{\tilde{l}} - \mathbf{V}_l\right)\nabla w_{\tilde{l}} + \left(\mathbf{V}_l - \mathbf{V}_{\tilde{l}-1}\right)\nabla w_{\tilde{l}-1}\right]$$

For the determination of this expression at the outest levels, the above free-slip conditions on $\mathbf{V}$ are used.

The same approach is applied for the term involving $[(\partial\mathbf{V}/\partial\eta).\nabla\phi]$ in (2.62).

4.10 Conservation of Energy by the Vertical Scheme

The conserved total energy for the continuous equations is given by (2.50). Following Simmons and Burridge (1981), the preservation by the finite-difference scheme of energy conservation is achieved if, for any variable F, the vertical integral of the advection term verifies the $per\ partes$ following property, given the fact that $\dot{\eta} = 0$ at the upper and lower boundaries of the domain:

$$\begin{aligned}
\int_0^1 m\dot{\eta}\frac{\partial F}{\partial \eta}d\eta &= -\int_0^1 F\frac{\partial}{\partial \eta}(m\dot{\eta})d\eta \\
\int_0^1 m\dot{\eta}F\frac{\partial F}{\partial \eta}d\eta &= -\int_0^1 \frac{F^2}{2}\frac{\partial}{\partial \eta}(m\dot{\eta})d\eta
\end{aligned}$$

The discrete counterpart of this property is true if the vertical advection term is chosen as:

$$\left(\dot{\eta}\frac{\partial F}{\partial \eta}\right)_l = \frac{1}{2\delta\pi_l}\left[(m.\dot{\eta})_{\tilde{l}}\left(F_{l+1} - F_l\right) + (m.\dot{\eta})_{\tilde{l}-1}\left(F_l - F_{l-1}\right)\right]$$

The conservation of the total energy by the vertical advective scheme thus leads to an imposed form for the vertical advection terms. All this discussion appears to be independant of wether the system is hydrostatic or non-hydrostatic, hence all these results apply identically to the Euler equations system.

Finally, the form of advective terms does not need to be changed with respect to the form in the hydrostatic primitive equation model.

4.11 Conservation of Angular Momentum by the Vertical Scheme

The conservation of the total angular momentum is the consequence of a stronger property (2.52). This property will be used as a constraint to determine the discretised form of the hydrostatic part of the horizontal pressure-gradient force.

4.11.1 Constraint for the pressure gradient

We introduce here a definition for the discretised version of $(\partial p/\partial \eta)$ at level l:

$$\left(\frac{\partial p}{\partial \eta}\right)_l = \frac{(\delta p)_l}{\delta \eta_l}$$

The finite-difference version of the constraint (2.52) writes:

$$\begin{aligned}
\sum_{l=1}^{L}\left(RT\frac{\nabla p}{p}\right)_l \delta\pi_l &= \sum_{l=1}^{L}\left(\phi_l - \phi_s\right)\nabla(\delta p)_l \\
&= \sum_{l=1}^{L}\left(\alpha_l\frac{R_lT_l}{1+\mathcal{P}_l} + \sum_{k=l+1}^{L}\frac{R_kT_k\delta_k}{1+\mathcal{P}_k}\right)\nabla(\delta p)_l \\
&= \sum_{l=1}^{L}\alpha_l\frac{R_lT_l}{1+\mathcal{P}_l}\nabla\left(\delta p\right)_l + \sum_{k=1}^{L}\left(\frac{R_kT_k}{1+\mathcal{P}_k}\delta_k\sum_{l=1}^{k-1}\nabla(\delta p)_l\right)
\end{aligned}$$

This leads to the discrete form of the non-hydrostatic pressure gradient imposed by the conservation of angular momentum:

$$\left(RT\frac{\nabla p}{p}\right)_l = \frac{R_lT_l}{1+\mathcal{P}_l}\frac{1}{\delta\pi_l}\left(\alpha_l\nabla\left(\delta p\right)_l + \delta_l\sum_{k=1}^{l-1}\nabla(\delta p)_k\right) \tag{4.27}$$

The continuous constraint thus imposes the form of $(\delta p)_l$: In the fully compressible system described here, the horizontal pressure gradient must not depend on the state of the atmosphere at others levels. Hence $(\delta p)_l$ $must$ have the form of a difference in order to insure cancellations trough vertical summations in the right hand side term. Consequently, we can specify the following form for $(\delta p)_l$:

$$(\delta p)_l = p_{\tilde{l}} - p_{\tilde{l}-1}$$

Any discretisation of $(\partial p/\partial \eta)$ which respects this form will insure the conservation of angular momentum and locality of the non-hydrostatic pressure gradient. The last stage is thus to choose a discrete form for $p_{\tilde{l}}$. Starting from $p = \pi + \pi\mathcal{P}$, and according to the fact that the prognostic variable is $\mathcal{P}$, the most "natural" form for discretising $p_{\tilde{l}}$ would be to express $\mathcal{P}$ at interfaces (which is noted $\bar{\mathcal{P}}$) as an average, defining:

$$\bar{\mathcal{P}}_{\tilde{l}} = \left(\frac{\mathcal{P}_{l+1} + \mathcal{P}_l}{2}\right)$$

However, this would create a mismatch between the $\nabla\mathcal{P}$ operator in the linear system and the $\nabla\mathcal{P}$ operator in the complete system. This mismatch has been shown to result in a very unstable behaviour, and hence cannot be accepted. We are facing here an example where the fulfilment of a relatively minor constraint violates a major one.

As a consequence, it has been decided to relax the constraint of discrete angular-momentum conservation for the non-hydrostatic part of the pressure force, and to insure this conservation only for the hydrostatic part.

Hence we separate the hydrostatic and non-hydrostatic parts of $(RT\nabla p/p)$:

$$RT\frac{\nabla p}{p} = RT\frac{\nabla \pi}{\pi} + \frac{RT}{1+\mathcal{P}}\frac{\nabla \mathcal{P}}{\mathcal{P}}$$

The last RHS term is not subjected to the angular momentum constraint and is discretized in the natural way. The first RHS term is discretised by fulfilling the angular momentum constraint. hence the discretisation of the total pressure term is:

$$\left(RT\frac{\nabla p}{p}\right)_l = R_l T_l \frac{1}{\delta\pi_l}\left(\alpha_l \nabla\left(\delta\pi\right)_l + \delta_l \sum_{k=1}^{l-1} \nabla(\delta\pi)_k\right) + \frac{R_l T_l}{1+\mathcal{P}_l}\nabla\mathcal{P}_l \tag{4.28}$$

and consequently, exploiting $\delta\pi_l = \pi_{\tilde{l}} - \pi_{\tilde{l}-1}$:

$$\left(\frac{RT}{p}\nabla p\right)_l = R_l T_l \frac{1}{\delta\pi_l}\left(\alpha_l \nabla\left(\delta\pi_l\right) + \delta_l \nabla\pi_{\tilde{l}-1}\right) + \frac{R_l T_l}{1+\mathcal{P}_l}\nabla\mathcal{P}_l \tag{4.29}$$

Using the coordinate definition, the hydrostatic part can be simplified into:

$$R_l T_l \frac{1}{\delta\pi_l}\left(\alpha_l \nabla\delta\pi_l + \delta_l \nabla\pi_{\tilde{l}-1}\right) = R_l T_l \frac{1}{\delta\pi_l}\left(\frac{\delta_l}{\delta\pi_l}C_l + \delta B_l\right)\nabla\pi_s$$

with:

$$C_l = A_{\tilde{l}}B_{\tilde{l}-1} - A_{\tilde{l}-1}B_{\tilde{l}}$$

This form is equivalent to the form found for $(RT\nabla\pi/\pi)$ in the hydrostatic case (see Simmons and Burridge, 1981).

4.11.2 Expression of the $\nabla\phi$ term

The discretised diagnostic relation for the geopotential is:

$$\phi_l = \phi_s + \sum_{k=l+1}^{L} \left(\frac{R_k T_k}{1+\mathcal{P}_k} \delta_k \right) + \alpha_l \frac{R_l T_l}{1+\mathcal{P}_l}$$

The gradient then writes:

$$\begin{aligned}
(\nabla\phi)_l &= \nabla\phi_s + \sum_{k=l+1}^{L} \left[\frac{R_k T_k}{1+\mathcal{P}_k} \nabla\delta_k + \frac{\nabla(RT)_k}{1+\mathcal{P}_k}\delta_k - \frac{\delta_k}{(1+\mathcal{P}_k)} \left(\frac{RT}{1+\mathcal{P}} \nabla\mathcal{P} \right)_k \right] \\
&\quad + \frac{\alpha_l}{1+\mathcal{P}_l} \nabla(RT)_l + \frac{R_l T_l}{1+\mathcal{P}_l} \nabla\alpha_l - \frac{\alpha_l}{(1+\mathcal{P}_l)} \left(\frac{RT}{1+\mathcal{P}} \nabla\mathcal{P} \right)_l
\end{aligned}$$

The horizontal gradients of α and δ, can also be expressed in term of hydrostatic surface pressure horizontal gradient after some algebra:

$$\begin{aligned}
\nabla\alpha_l &= -\frac{1}{2}.\frac{(1-\alpha_l)C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}} \nabla\pi_s \\
\nabla\delta_l &= -\frac{1}{2}.\frac{C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}} \frac{(1-\alpha_l)^2+1}{(1-\alpha_l)} \nabla\pi_s
\end{aligned}$$

Chapter 5

Time Discretisation

(19/11/2003)

5.1 Introduction

The discretisation in time of meteorological models is a vast subject, and many types of time-schemes have been designed among meteorological services. In Aladin-NH, several time-schemes based on similar ideas are implemented, although only one is to be used operationally. This chapter describes the most important time-discretisation aspects from the point of view of the NH version and of the future operational use. It is important to note that except the solution of the linear implicit problem (section 5.10), and the special time-discretisation of the so-called "X-term" (section 5.8), the details of the time-discretisation are not specific to the NH version: they are designed in exactly the same way as for the HPE version. A more extensive documentation on less important variants of the possible time-discretisations of Aladin/ARPEGE can be found in the official documentation of ARPEGE.

Time schemes (SI and ICI):

Partially-implicit schemes allow large time-steps compared to explicit time-schemes, and thus they result in a better efficiency. As outlined in section 3.2, all the time-schemes of Aladin-NH are based on the idea of a constant-coefficient linear separation of the source terms of the evolution system, and a partially-implicit treatment based on this constant-coefficients linearisation.

Two types of partly implicit schemes are implemented in Aladin-NH: semi-implicit (SI) and iterative-centred-implicit (ICI). The ICI scheme is an iterative scheme which aim is to approach the fully-implicit-centred (FIC) scheme. The FIC scheme is generally believed to be optimally robust and accurate. The ICI scheme is sometimes (abusively) referred to as "predictor-corrector" (PC) scheme, but this terminology should not be used because it is misleading.

Marching schemes (3-TL and 2-TL):

Three main types of time-marching schemes are implemented in Aladin-NH:
- 3-time-levels centred scheme (3-TL, also referred to as "leap-frog")
- 2-time-levels extrapolating centred scheme (2-TL E)
- 2-time-levels non-extrapolating scheme (2-TL NE)

Transport schemes (Eul and SL):

Although it may seem surprising, the Eulerian (Eul) or semi-Lagrangian (SL) treatment of material derivatives has many links with the time-discretisation. In particular various time-discretisations can be used only with a SL transport scheme.

Forbidden combinations:

Some combinations of time-schemes, marching-schemes, and transport schemes are not relevant: The SI scheme should not be used with a 2-TL NE marching-scheme, because this combination is not second-order accurate in time (only first-order accurate). The combination of an ICI scheme with a 3-TL marching-scheme is not very interesting because for 3-TL schemes, 2 iterations are needed and the resulting scheme is thus not very efficient. Finally, 2-TL schemes are valid only for the SL transport-scheme because the advective terms of the Eulerian version are unstable when treated in a 2-TL way.

Summary of the combinations described below:

As a consequence of the latter remarks, the description in this documentation is limited to the following list:
- 3-TL SI (SL and Eul) schemes
- 2-TL E SI SL scheme
- 2-TL NE ICI SL scheme

5.2 Symbolic notation for time discretisations

We now introduce a symbolic notation for the explanation of the principle of the time schemes of Aladin-NH. In effect it should be outlined that the choice and the form of a time discretisation is largely independent of the details of the evolution system by itself. In other words the basic principle of a given time-scheme is exactly the same for a shallow-water system than for a EE system. Moreover, for the description of the principle of a time-scheme, it is not very important to know if the system is space-discretised or space-continuous. Hence, since the nature of the evolution system is not of great importance for presenting the principle of the time-discretisations, the equations of the complete system, the linear implicit operator, and the state variable do not need to be expanded; they can be condensed in a symbolic notation.
The symbolic notation used in this chapter to describe the time-discretisation is valid for the space-discretised model as well as for an idealised space-continuous model. This latter feature is extensively used for also condensing the algebra in theoretical analyses.

Symbolically, the state of the atmosphere can be represented by $\mathcal{X}$. For the space-discretised model $\mathcal{X}$ is a vector containing the whole grid-point values or spectral-coefficients of all prognostic variables on the domain at a given time. For a space-continuous model, $\mathcal{X}$ would be a vector of continuous functions defined in the atmospheric domain, and the size of the vector is then the number of prognostic variables.
The whole model evolution system applying to $\mathcal{X}$ is noted $\mathcal{M}$, which e.g. represents the RHS of the systems (2.54) – (2.58). or (2.66) – (2.70) in the space-continuous case.
In the time-continuous form, the original system (2.54) – (2.58) thus symbolically writes:

$$\frac{\partial \mathcal{X}}{\partial t} = \mathcal{M}.\mathcal{X}$$

Similarly, the linear operator in the RHS of (3.14) – (3.18) is noted $\mathcal{L}$. The time-continuous evolution of a pertubation $\mathcal{X}'$ by this linear operator is thus:

$$\frac{\partial \mathcal{X}'}{\partial t} = \mathcal{L}.\mathcal{X}'$$

In this chapter, the following notations are used:

$$
\begin{aligned}
\mathcal{X}^{+} &= \mathcal{X}(t+\Delta t) \\
\mathcal{X}^{-} &= \mathcal{X}(t-\Delta t) \\
\mathcal{X}^{0} &= \mathcal{X}(t) \\
\mathcal{X}_F &= \mathcal{X}(F) \qquad \text{at the final point of the SL trajectory} \\
\mathcal{X}_O &= \mathcal{X}(O) \qquad \text{at the origin point of the SL trajectory (interpolated)} \\
\mathcal{I} &= \text{the identity operator (in any space)}
\end{aligned}
$$

5.3 3-TL SI SL scheme

The position of the origin point O has first to be found. This is done by solving the equation:

$$OF = 2\Delta t V^0$$

where V^0 is the wind at time t.
In the 3-TL SI SL time scheme, the evolution of the model is discretised as follows (see Tanguay, 1992):

$$\frac{\mathcal{X}_F^+ - \mathcal{X}_O^-}{2\Delta t} = \frac{\mathcal{M}.\mathcal{X}_F^0 + \mathcal{M}.\mathcal{X}_O^0}{2} + \mathcal{L}\left(\frac{\mathcal{X}_F^+ + \mathcal{X}_O^-}{2} - \frac{\mathcal{X}_F^0 + \mathcal{X}_O^0}{2}\right) \tag{5.1}$$

$\Rightarrow$

$$\mathcal{X}_F^+ = \mathcal{X}_O^- + \Delta t\,(\mathcal{M}.\mathcal{X}_F^0 + \mathcal{M}.\mathcal{X}_O^0) + \Delta t\,\mathcal{L}\left(\mathcal{X}_F^+ + \mathcal{X}_O^- - \mathcal{X}_F^0 + \mathcal{X}_O^0\right) \tag{5.2}$$

Moving all the terms at $(t+\Delta t)$ to the left hand side and dropping the subscript F, the evolution can be written as a linear sytem which to be solved for $\mathcal{X}^+$:

$$\left(\mathcal{X}^+ - \Delta t\,\mathcal{L}\mathcal{X}^+\right) = \underbrace{\mathcal{X}_O^- + \Delta t\,(\mathcal{M}.\mathcal{X}_F^0 + \mathcal{M}.\mathcal{X}_O^0)}_{\mathcal{X}^+_{\mathsf{NL}}} + \underbrace{\Delta t\,\mathcal{L}\left(\mathcal{X}_O^- - (\mathcal{X}_F^0 + \mathcal{X}_O^0)\right)}_{\delta\mathcal{X}^+_{\mathsf{Lin}}} \tag{5.3}$$

In the right hand side, the first bracketed term is traditionally called "explicit guess" since this is the value that would be obtained with an explicit leap-frog scheme, and the second one is traditionally called "explicit part of the semi-implicit correction". However, in this documentation, we use a notation $\delta\mathcal{X}^+_{\mathsf{Lin}}$ for the part which only involves the linear operator, and $\mathcal{X}^+_{\mathsf{NL}}$ for the remaining part.
In the Aladin-NH model, these two terms are computed separately, then gathered, in the so-called "RHS of the implicit system", which is noted $\widetilde{\mathcal{X}^+}$:

$$\widetilde{\mathcal{X}^+} = \mathcal{X}^+_{\mathsf{NL}} + \delta\mathcal{X}^+_{\mathsf{Lin}} \tag{5.4}$$

The completion of the semi-implicit scheme then consists in inverting the following linear system:

$$(\mathcal{I} - \Delta t\,\mathcal{L})\,\mathcal{X}^+ = \widetilde{\mathcal{X}^+} \tag{5.5}$$

This problem is decomposed for each spectral coefficient and variable, under the form of vertical columns, as it will be depicted in following sections.

5.4 3-TL SI Eulerian scheme

For the Eulerian 3-TL SI section, the latter version remains valid except that all subscripts F and O are removed (the model $\mathcal{M}$ thus additionally contains advection terms, and interpolations are no longer used).

5.5 2-TL E SI SL schemes

In a general way, 2-TL schemes are roughly twice more efficient than 3-TL schemes, because a simple temporal "transfer function" is used instead of a full-model integration, to determine the intermediate state along the trajectories. This is the reason why 2-TL SI schemes are used for the operational HPE systems ARPEGE/IFS and Aladin.
As outlined above, two main types of 2-TL schemes can be distinguished: 2-TL extrapolating (2-TL E) schemes, and 2-TL non-extrapolating (2-TL NE) schemes, but the combination 2-TL NE SI is forbidden, because it is only first-order accurate in time. Hence, the description is restricted here to 2-TL E SI schemes.
In Aladin-NH, there are two variants of the 2-TL E SI scheme: namely the "normal extrapolation" scheme and the "stable extrapolation" scheme (called "SETTLS"). These two variants only differ by the way the time-extrapolation is performed. The "SETTLS" extrapolation is more stable than the normal one, even for the HPE system. The "SETTLS" is more consistent than the "normal" extrapolating scheme. Hence, in this documentation, only the "SETTLS" scheme is described.
A last remark concerning 2-TL E SI schemes is that they are "not really" 2-TL schemes because the extrapolation necessarily involves a third time-level. However, their behaviour is much closer to 2-TL schemes than 3-TL schemes, and this is why they are usually categoricized in the class of 2-TL schemes.

The position of the origin point O has first to be found. This is done by solving (in O) the equation:

$$OF = 0.5\Delta t(V_F^0 + 2V_O^0 - V_O^-)$$

The evolution of the model is then discretised as follows:

$$\begin{aligned}\frac{\mathcal{X}_F^+ - \mathcal{X}_O^0}{\Delta t} &= \frac{\mathcal{M}.\mathcal{X}_F^0 + 2\mathcal{M}.\mathcal{X}_O^0 - \mathcal{M}.\mathcal{X}_O^-}{2} \\ &+ \mathcal{L}\left(\frac{\mathcal{X}_F^+ + \mathcal{X}_O^0}{2} - \frac{\mathcal{X}_F^0 + 2\mathcal{X}_O^0 - \mathcal{X}_O^-}{2}\right)\end{aligned} \tag{5.6}$$

$\Rightarrow$

$$\begin{aligned}\mathcal{X}_F^+ &= \mathcal{X}_O^0 + \frac{\Delta t}{2}\left(\mathcal{M}.\mathcal{X}_F^0 + 2\mathcal{M}.\mathcal{X}_O^0 - \mathcal{M}.\mathcal{X}_O^-\right) \\ &+ \frac{\Delta t}{2}\mathcal{L}\left[(\mathcal{X}_F^+ + \mathcal{X}_O^-) - (\mathcal{X}_F^0 + \mathcal{X}_O^0)\right]\end{aligned} \tag{5.7}$$

Grouping the $\mathcal{X}^+$ terms at left, and dropping the subscript "F" yields:

$$\begin{aligned}\left(\mathcal{X}^+ - \frac{\Delta t}{2}\mathcal{L}\mathcal{X}^+\right) &= \overbrace{\mathcal{X}_O^0 + \frac{\Delta t}{2}\left(\mathcal{M}.\mathcal{X}_F^0 + 2\mathcal{M}.\mathcal{X}_O^0 - \mathcal{M}.\mathcal{X}_O^-\right)}^{\mathcal{X}_{\mathsf{NL}}^+} \\ &+ \underbrace{\frac{\Delta t}{2}\mathcal{L}\left[\mathcal{X}_O^- - (\mathcal{X}_F^0 + \mathcal{X}_O^0)\right]}_{\delta\mathcal{X}_{\mathsf{Lin}}^+}\end{aligned} \tag{5.8}$$

Similarly to above, we have:

$$\widetilde{\mathcal{X}^+} = \mathcal{X}_{\mathsf{NL}}^+ + \delta\mathcal{X}_{\mathsf{Lin}}^+ \tag{5.9}$$

The completion of the semi-implicit scheme then consists in inverting the following linear system:

$$\left(\mathcal{I} - \frac{\Delta t}{2}\mathcal{L}\right)\mathcal{X}^+ = \widetilde{\mathcal{X}^+} \tag{5.10}$$

5.6 2-TL NE ICI SL schemes

Even if 2-TL SI schemes can be used without particular problem at a resolution of 2 km, explicitly treated non-linear terms may result in a lack of robustness for higher resolutions (and thus steeper orography). This is why a 2-TL ICI scheme is implemented in Aladin-NH, in order to guarantee an optimal robustness of the model even at higher resolutions. The target scheme for an operational use in 2008 in AROME is a 2-TL ICI scheme.
In ICI schemes, the non-extrapolating option can legitimately be used because the iteration restores to second-order in time accuracy, as soon as at least one iteration is used. As mentionned above, ICI schemes are based on the iterative solution of the non-linear implicit problem resulting from a FIC scheme. The iteration index is noted (i).

The position of the origin point $O^{(0)}$ is found by solving the equation:

$$O^{(0)}F = 0.5\Delta t(V_F^0 + V_{O^{(0)}}^0)$$

Then, a 2TL NE SI time-step is performed, and the result is noted $\mathcal{X}^{+(0)}$:

$$\frac{\mathcal{X}_F^{+(0)} - \mathcal{X}_{O^{(0)}}^0}{\Delta t} = \frac{\mathcal{M}.\mathcal{X}_F^0 + \mathcal{M}.\mathcal{X}_{O^{(0)}}^0}{2} + \mathcal{L}\left(\frac{\mathcal{X}_F^{+(0)} - \mathcal{X}_{O^{(0)}}^0}{2}\right) \tag{5.11}$$

Then, the following algorithm is iterated:

$$O^{(i)}F = 0.5\Delta t(V_F^{+(i-1)} + V_{O^{(i)}}^0)$$

$$\frac{\mathcal{X}_F^{+(i)} - \mathcal{X}_{O^{(i)}}^0}{\Delta t} = \frac{\mathcal{M}.\mathcal{X}_F^{+(i-1)} + \mathcal{M}.\mathcal{X}_{O^{(i)}}^0}{2} + \mathcal{L}\left(\frac{\mathcal{X}_F^{+(i)} - \mathcal{X}_{O^{(i)}}^{+(i-1)}}{2}\right) \tag{5.12}$$

for $i = 1, \cdots, \texttt{NSITER}$. The final state $\mathcal{X}^+$ is then taken as the last iterated value $\mathcal{X}^{+(\texttt{NSITER})}$:

$$\mathcal{X}^+ = \mathcal{X}^{+(\texttt{NSITER})}$$

In practice, it is expected that one iteration $\texttt{NSITER} = 1$ is enough to maintain a satisfactory robustness at high resolutions.

Both equations (5.11), (5.12) can be rewritten concisely as:

$$\left(\mathcal{I} - \frac{\Delta t}{2}\,\mathcal{L}\right)\mathcal{X}^{+(i)} = \widetilde{\mathcal{X}^{+(i)}} \tag{5.13}$$

where:

$$\widetilde{\mathcal{X}^{+(i)}} = \mathcal{X}_{\mathsf{NL}}^{+(i)} + \delta\mathcal{X}_{\mathsf{Lin}}^{+(i)} \tag{5.14}$$

where:

$$\begin{aligned}
\mathcal{X}_{\mathsf{NL}}^{+(0)} &= \mathcal{X}_{O^{(0)}}^0 + \frac{\Delta t}{2}\left(\mathcal{M}.\mathcal{X}_F^0 + \mathcal{M}.\mathcal{X}_{O^{(0)}}^0\right) && (5.15)\\
\delta\mathcal{X}_{\mathsf{Lin}}^{+(0)} &= -\frac{\Delta t}{2}\mathcal{L}.\mathcal{X}_{O^{(0)}}^0 && (5.16)\\
\mathcal{X}_{\mathsf{NL}}^{+(i)} &= \mathcal{X}_{O^{(i)}}^0 + \frac{\Delta t}{2}\left(\mathcal{M}.\mathcal{X}_F^{+(i-1)} + \mathcal{M}.\mathcal{X}_{O^{(i)}}^0\right) && (5.17)\\
\delta\mathcal{X}_{\mathsf{Lin}}^{+(i)} &= -\frac{\Delta t}{2}\mathcal{L}.\mathcal{X}_{O^{(i)}}^{+(i-1)} && (5.18)
\end{aligned}$$

$\mathcal{X}_{\mathsf{Lin}}$ is the part involving $\mathcal{L}$ in the RHS of (5.11)-(5.12), while $\mathcal{X}_{\mathsf{NL}}$ is the remaining part.
Equations (5.13) and (5.14) are valid for $i = 0, \cdots, \texttt{NSITER}$.

5.7 Choice of prognostic variables

It has been shown in Bénard (2003), Bénard et al. (2004), and Bénard et al. (2005) that the choice of the prognostic variables may have a tremendous impact on the stability of the SI or ICI scheme of the NH model.

*** TO BE WRITTEN ***

5.8 Special explicit treatment of the X-term for the $\mathbb{d}$ variable

In the case where the $\mathbb{d}$ variable is used, it has been mentionned in Chapter 2, that the evolution of the so-called X-term in the RHS of (2.47) was introduced through a specific treatment. This special treatment is detailed in the present section.

The equation (2.47) is rewritten for conciseness as follows:

$$\frac{d\mathbb{d}}{dt} = \mathsf{RHS}_{\mathsf{d}} + \dot{\mathsf{X}} \tag{5.19}$$

The problem is to evaluate $\dot{\mathsf{X}}$. An explicit evaluation by expanding X in terms of the basic variables T, $\mathcal{P}$, π, etc. would lead to a cumbersome equation. In Aladin-NH (similarly to MC2 and CRCM canadian models) it has been chosen to replace this cumbersome evolution group by an explicit evaluation along the SL trajectory, using only known past information. This approach is detailed hereafter for each time-scheme of Aladin-NH (the discretisation of $\mathsf{RHS}_{\mathsf{d}}$ is not expanded, for clarity):

3-TL SL SI scheme

$$\mathbb{d}_F^+ = \mathbb{d}_O^- + 2\Delta t\ \mathsf{RHS}_{\mathsf{d}} + (\mathsf{X}_F^0 + \mathsf{X}_O^0 - 2\mathsf{X}_O^-) \tag{5.20}$$

3-TL Eul SI scheme

$$\mathbb{d}^+ = \mathbb{d}^- + 2\Delta t\ \mathsf{RHS}_{\mathsf{d}} + 2(\mathsf{X}^0 - \mathsf{X}^-) \tag{5.21}$$

2-TL SI schemes (and first iteration of the 2-TL NE ICI scheme)

$$\mathbb{d}_F^+ = \mathbb{d}_O^0 + \Delta t\ \mathsf{RHS}_{\mathsf{d}} + (\mathsf{X}_F^m + \mathsf{X}_O^m - 2\mathsf{X}_O^0) \tag{5.22}$$

where X^m denotes extrapolated (or not) values, according to the type of extrapolation which is chosen ("SETTLS" extrapolation, or no extrapolation). This expression is also valid for the initial evaluation [denoted by $i = (0)$ in (5.11)] of the 2-TL NE ICI scheme.

subsequent iterations $(i) \geq 1$ of the 2-TL NE ICI scheme

$$\mathbb{d}_F^{+(i)} = \mathbb{d}_O^0 + \Delta t\ \mathsf{RHS}_{\mathsf{d}} + (\mathsf{X}_F^{+(i-1)} - \mathsf{X}_O^0) \tag{5.23}$$

We note that these special treatments are not second-order accurate in time for SI schemes. However, as soon as at least one ICI iteration is performed $[(i) \geq 1]$, the second-order accuracy in time is restored.

5.9 Rigid bottom boundary condition in the SL context

It has been seen in section 4.7.2 that for SL schemes, the rigid bottom boundary condition could be expressed directly in terms of the parcels' vertical acceleration $\dot{w}_s$ We may write, for a 3-TL scheme:

$$\dot{w}_s = \frac{w_{sF}^{+*} - w_{sO}^{-}}{2\Delta t} \tag{5.24}$$

In this equation, w_{sF}^{+*} is an explicit guess of the surface vertical velocity at time $(t + \Delta t)$. In the current version, this guess is given by the following equation:

$$\frac{w_{sF}^{+*} - w_{sO}^{-}}{2\Delta t} = \frac{\mathcal{M}_F^0 + \mathcal{M}_O^0}{2} + \frac{\mathcal{L}_O^0 + \mathcal{L}_O^-}{2} - \mathcal{L}_O^0 \tag{5.25}$$

where $\mathcal{M}$ are the sources of w in the full model, and $\mathcal{L}$ are the sources of w in the linear SI model. These sources are deduced from the sources for the vertical momentum variable d or $\mathbb{d}$.

The extension to 2-TL SL schemes is straightforward, and is not developed here.

5.10 Solution of the implicit problem

In all cases (3-TL SI, 2-TL SI, and 2-TL ICI), we see that the implicit problem to be solved is the same, and writes symbolically:

$$(\mathcal{I} - \delta t\, \mathcal{L})\, \mathcal{X} = \widetilde{\mathcal{X}} \tag{5.26}$$

where the unknown is $\mathcal{X}$ and the RHS $\widetilde{\mathcal{X}}$ is known explicitly. The precise meaning of δt and $\mathcal{X}$ depends of the type of scheme:

$$\begin{aligned} \text{3-TL schemes} \quad &\longrightarrow \quad \delta t = \Delta t \\ \text{2-TL schemes} \quad &\longrightarrow \quad \delta t = (\Delta t/2) \end{aligned}$$

and:

$$\begin{aligned} \text{SI schemes} \quad &\longrightarrow \quad \mathcal{X} = \mathcal{X}^{+} \\ \text{ICI scheme (index } i\text{)} \quad &\longrightarrow \quad \mathcal{X} = \mathcal{X}^{+(i)} \end{aligned}$$

The system (5.26) is now cast in spectral space for each individual variable column at spectral location (m, n). The variables are now considered as being column vectors of vertical values from level 1 to L for a given spectral component (m, n). Similarly, the linear operators are considered as being matrices applying to these vectors. The vertical column vector corresponding to a variable ψ is noted $\underline{\psi}$ (and similarly $\underline{\pi_s} = \pi_s.(1, 1, ...1)$). Substituting the actual expression of $\mathcal{L}$ given by (3.27)–(3.31) in the symbolic equation for the implicit problem (5.26) yields:

$$\begin{aligned} \underline{D'} - \delta t \Delta' \left[RT^* \left(\mathbf{G}^* - 1\right) \underline{\mathcal{P}} - R\mathbf{G}^* \underline{T} - \frac{RT^*}{\pi_s^*} \underline{\pi_s} \right] &= \underline{\widetilde{D}} \\ \underline{\mathsf{d}} - \delta t \left(-\frac{g}{rH_*} \mathbf{L}^* \underline{\mathcal{P}} \right) &= \underline{\widetilde{\mathsf{d}}} \\ \underline{\mathcal{P}} - \delta t \left[\left(\mathbf{S}^* - \frac{C_p}{C_v} \right) \mathbf{m}_*^2 \underline{D'} - \frac{C_p}{C_v} \underline{\mathsf{d}} \right] &= \underline{\widetilde{\mathcal{P}}} \end{aligned}$$

$$\begin{aligned}
\underline{T} - \delta t \left(-\frac{RT^*}{C_v} \mathbf{m}_*^2 \underline{D'} - \frac{RT^*}{C_v} \underline{\mathsf{d}} \right) &= \underline{\widetilde{T}} \\
\underline{\pi_s} - \delta t \left(-\pi_s^* \mathbf{N}^* \mathbf{m}_*^2 \underline{D'} \right) &= \underline{\widetilde{\pi_s}}
\end{aligned}$$

As an illustration, for a 3-TL SI Eul scheme we would have:

$$\begin{aligned}
\underline{\widetilde{D}} &= \frac{1}{\mathbf{m}^2} \underline{D_{\mathsf{NL}}^+} + \frac{1}{\mathbf{m}^2} \underline{\delta D_{\mathsf{Lin}}^+} \\
\underline{\widetilde{\mathsf{d}}} &= \underline{\mathsf{d}_{\mathsf{NL}}^+} + \underline{\delta \mathsf{d}_{\mathsf{Lin}}^+} \\
\underline{\widetilde{\mathcal{P}}} &= \underline{\mathcal{P}_{\mathsf{NL}}^+} + \underline{\delta \mathcal{P}_{\mathsf{Lin}}^+} \\
\underline{\widetilde{T}} &= \underline{T_{\mathsf{NL}}^+} + \underline{\delta T_{\mathsf{Lin}}^+} \\
\underline{\widetilde{\pi_s}} &= \underline{\pi_{s\,\mathsf{NL}}^+} + \underline{\delta \pi_{s\,\mathsf{Lin}}^+}
\end{aligned}$$

and (still for a 3-TL Eul scheme only):

$$\begin{aligned}
\underline{\delta D_{\mathsf{Lin}}^+} &= \delta t \Delta \left[RT^* \left(\mathbf{G}^* - 1 \right) \left(\underline{\mathcal{P}}^- - 2\underline{\mathcal{P}}^0 \right) - R\mathbf{G}^* \left(\underline{T}^- - 2\underline{T}^0 \right) - \frac{RT^*}{\pi_s^*} \left(\underline{\pi_s}^- - 2\underline{\pi_s}^0 \right) \right] \\
\underline{\delta \mathsf{d}_{\mathsf{Lin}}^+} &= \delta t \left(-\frac{g}{H} \mathbf{L}^* \right) \left(\underline{\mathcal{P}}^- - 2\underline{\mathcal{P}}^0 \right) \\
\underline{\delta \mathcal{P}_{\mathsf{Lin}}^+} &= \delta t \left[\left(\mathbf{S}^* - \frac{C_p}{C_v} \right) \frac{\mathbf{m}_*^2}{\mathbf{m}^2} \left(\underline{D}^- - 2\underline{D}^0 \right) - \frac{C_p}{C_v} \left(\underline{\mathsf{d}}^- - 2\underline{\mathsf{d}}^0 \right) \right] \\
\underline{\delta T_{\mathsf{Lin}}^+} &= \delta t \left[-\frac{RT^*}{C_v} \frac{\mathbf{m}_*^2}{\mathbf{m}^2} \left(\underline{D}^- - 2\underline{D}^0 \right) - \frac{RT^*}{C_v} \left(\underline{\mathsf{d}}^- - 2\underline{\mathsf{d}}^0 \right) \right] \\
\underline{\delta \pi_{s\,\mathsf{Lin}}^+} &= \delta t \left[-\pi_s^* \mathbf{N}^* \frac{\mathbf{m}_*^2}{\mathbf{m}^2} \left(\underline{D}^- - 2\underline{D}^0 \right) \right]
\end{aligned}$$

The extension to other cases (2-TL E SI SL, and 2-TL NE ICI SL) is straightforward using the form of the RHS in (5.8) and (5.15)–(5.18).

Note that the computation of tilded quantities being made in grid point space, they use only physical quantities. Moreover, the computation of $\underline{\widetilde{D}}$ is in fact decomposed in the code, exactely as in the hydrostatic version, by first computing $\underline{\widetilde{u'}}$ and $\underline{\widetilde{v'}}$ in grid point space and then combining them during spectral transforms as mentionned above. We won't go into more details about this subject since it is not a non-hydrostatic specificity.

5.10.1 Elimination of variables

The elimination process is similar to the one used for the derivation of the structure equation of the linear system. The $\underline{\mathsf{d}}$ and $\underline{D'}$ variables are first kept:

$$\begin{aligned}
\underline{\mathsf{d}} &= \underline{\mathsf{d}^\bullet} + \delta t^2 \left[\left(-\frac{g}{rH_*} \mathbf{L}^* \right) \left(\mathbf{S}^* - \frac{C_p}{C_v} \right) \mathbf{m}_*^2 \underline{D'} + \left(-\frac{g}{rH_*} \mathbf{L}^* \right) \left(-\frac{C_p}{C_v} \right) \underline{\mathsf{d}} \right] \\
&= \underline{\mathsf{d}^\bullet} + \delta t^2 \left[\frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \underline{D'} + c_*^2 \frac{\mathbf{L}^*}{rH_*^2} \underline{\mathsf{d}} \right]
\end{aligned}$$

and

$$\begin{aligned}
\underline{D'} = \underline{D^\bullet} + \delta t^2 \Delta' \Bigg\{ &\left[RT^* \left(\mathbf{G}^* - 1 \right) \left(\mathbf{S}^* - \frac{C_p}{C_v} \right) + \left(-R\mathbf{G}^* \right) \left(-\frac{RT^*}{C_v} \right) + \right. \\
&\left. \left(-\frac{RT^*}{\pi_s^*} \right) \left(-\pi_s^* \mathbf{N}^* \right) \right] \mathbf{m}_*^2 \underline{D'} + \left[RT^* \left(\mathbf{G}^* - 1 \right) \left(-\frac{C_p}{C_v} \right) + \left(-R\mathbf{G}^* \right) \left(-\frac{RT^*}{C_v} \right) \right] \underline{\mathsf{d}} \Bigg\}
\end{aligned}$$

which yields, by virtue of (4.1):

$$\underline{D'} = \underline{D}^{\bullet} + \delta t^2 \Delta' \left[c_*^2 \mathbf{m}_*^2 \underline{D'} + \left(-RT^* \mathbf{G}^* + c_*^2 \right) \underline{\mathsf{d}} \right]$$

with the following explicit terms:

$$\begin{aligned} \underline{\mathsf{d}}^{\bullet} &= \underline{\widetilde{\mathsf{d}}} + \delta t \left[\left(-\frac{g}{rH_*} \mathbf{L}^* \right) \underline{\widetilde{\mathcal{P}}} \right] \\ \underline{D}^{\bullet} &= \underline{\widetilde{D}} + \delta t \Delta' \left[RT^* \left(\mathbf{G}^* - 1 \right) \underline{\widetilde{\mathcal{P}}} + \left(-R\mathbf{G}^* \right) \underline{\widetilde{T}} + \left(-\frac{RT^*}{\pi_s^*} \right) \underline{\widetilde{\pi_s}} \right] \end{aligned}$$

The linear system can be rearranged into:

$$\left(1 - \delta t^2 c_*^2 \frac{\mathbf{L}^*}{rH_*^2} \right) \underline{\mathsf{d}} = \underline{\mathsf{d}}^{\bullet} + \delta t^2 \frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \underline{D'} \tag{5.27}$$

$$\left(1 - \delta t^2 c_*^2 \Delta' \mathbf{m}_*^2 \right) \underline{D'} = \underline{D}^{\bullet} + \delta t^2 \Delta' \left(-RT^* \mathbf{G}^* + c_*^2 \right) \underline{\mathsf{d}} \tag{5.28}$$

We note that:

$$\begin{aligned} &\left(1 - \delta t^2 c_*^2 \mathbf{m}_*^2 \Delta' \right) . \left[\underline{\mathsf{d}}^{\bullet} + \delta t^2 \frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \underline{D'} \right] \\ &= \left(1 - \delta t^2 c_*^2 \mathbf{m}_*^2 \Delta' \right) \underline{\mathsf{d}}^{\bullet} + \left[\delta t^2 \frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \right] \left(1 - \delta t^2 c_*^2 \Delta' \mathbf{m}_*^2 \right) \underline{D'} \end{aligned}$$

Substituting (5.27) in the LHS and (5.28) in the last RHS term of the latter equation yields:

$$\begin{aligned} &\left(1 - \delta t^2 c_*^2 \mathbf{m}_*^2 \Delta' \right) \left(1 - \delta t^2 c_*^2 \frac{\mathbf{L}^*}{rH_*^2} \right) \underline{\mathsf{d}} \\ &= \left(1 - \delta t^2 c_*^2 \mathbf{m}_*^2 \Delta' \right) \underline{\mathsf{d}}^{\bullet} + \left[\delta t^2 \frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \right] \left[\underline{D}^{\bullet} + \delta t^2 \Delta' \left(-RT^* \mathbf{G}^* + c_*^2 \right) \underline{\mathsf{d}} \right] \\ &= \underline{\mathsf{d}}^{\bullet\bullet} + \left[\delta t^2 \frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \right] \left[\delta t^2 \Delta' \left(-RT^* \mathbf{G}^* + c_*^2 \right) \underline{\mathsf{d}} \right] \end{aligned}$$

with:

$$\underline{\mathsf{d}}^{\bullet\bullet} = \left(1 - \delta t^2 c_*^2 \mathbf{m}_*^2 \Delta' \right) \underline{\mathsf{d}}^{\bullet} + \delta t^2 \frac{\mathbf{L}^*}{rH_*^2} \left(-RT^* \mathbf{S}^* + c_*^2 \right) \mathbf{m}_*^2 \underline{D}^{\bullet} \tag{5.29}$$

By using the second discretisation constraint (4.4) in a similar way as in the time-continuous case, the expression for $\underline{\mathsf{d}}$ takes the final form:

$$\left[1 - \delta t^2 c_*^2 \left(\mathbf{m}_*^2 \Delta' + \frac{\mathbf{L}^*}{rH_*^2} \right) - \delta t^4 \frac{N_*^2 c_*^2}{r} \mathbf{m}_*^2 \Delta' \mathbf{T}^* \right] \underline{\mathsf{d}} = \underline{\mathsf{d}}^{\bullet\bullet} \tag{5.30}$$

5.10.2 Factorisation of the Helmholtz Equation spatial operator

The time- and space-discretised structure equation (5.30) is rewritten under a form consistent with the hydrostatic case, and readily exploitable in the case of a spatially-variable linearised map-factor $\mathbf{m}_*$:

$$\left[1 - \delta t^2 \mathbf{B}\, \mathbf{m}_*^2 \Delta' \right] \underline{\mathsf{d}} = \underline{\mathsf{d}}^{\bullet\bullet\bullet} \tag{5.31}$$

with:

$$\mathbf{B} = c_*^2 \left(1 - \delta t^2 c_*^2 \frac{\mathbf{L}^*}{rH_*^2}\right)^{-1} . \left(1 + \delta t^2 \frac{N_*^2}{r} \mathbf{T}^*\right) = \mathbf{T}_1^{-1}.\mathbf{T}_2 \tag{5.32}$$

and:

$$\underline{\mathsf{d}}^{\bullet\bullet\bullet} = \left(1 - \delta t^2 c_*^2 \frac{\mathbf{L}^*}{rH_*^2}\right)^{-1} .\underline{\mathsf{d}}^{\bullet\bullet} \tag{5.33}$$

and where $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined by:

$$\mathbf{T}_1 = \left(\mathbf{I} - \delta t^2 c_*^2 \frac{\mathbf{L}^*}{rH_*^2}\right) \tag{5.34}$$

$$\mathbf{T}_2 = c_*^2 \left(\mathbf{I} + \delta t^2 \frac{N_*^2}{r} \mathbf{T}^*\right) \tag{5.35}$$

It is seen that $\mathbf{T}_1$ and $\mathbf{T}_2$ are vertical tri-diagonal invertible and diagonalisable operators provided all the eigenvalues of $\mathbf{L}^*$ are negative with a multiplicity-order of one, and all the eigenvalues of $\mathbf{T}^*$ are positive with a multiplicity-order of one. These condition are found to be met for the finite-difference vertical discretisation described in Chapter 4.
Practically, the matrix $\mathbf{B}$ is computed, then diagonalised into:

$$\mathbf{B} = \mathbf{Q_B}.\Delta_\mathbf{B}.\mathbf{Q_B}^{-1} \tag{5.36}$$

These three matrices are computed in the model's setup and stored for use during the integration. The nonobvious code notation is :

$\mathbf{Q_B} =$ `SIMO`
$\mathbf{Q_B}^{-1} =$ `SIMI`
$\Delta_\mathbf{B} =$ `SIVP`

The RHS $\underline{\mathsf{d}}^{\bullet\bullet\bullet}$ is first decomposed in the basis of the eigenmodes of $\mathbf{B}$, i.e. $\mathbf{Q_B}^{-1}.\underline{\mathsf{d}}^{\bullet\bullet\bullet}$ is computed. For each eigenmode of $\mathbf{B}$ with index k, the implicit equation to be solved is a 2D equation which can be solved directly in the spectral space:

$$\left[\mathbf{Q_B}^{-1}.\underline{\mathsf{d}}\right]_k = \left[(1 - \delta t^2 \left[\delta_\mathbf{B}\right]_k \mathbf{m}_*^2 \Delta')\right]^{-1} \left[\mathbf{Q_B}^{-1}.\underline{\mathsf{d}}^{\bullet\bullet\bullet}\right]_k \tag{5.37}$$

where $[\delta_\mathbf{B}]_k$ is the diagonal element of $\Delta_\mathbf{B}$ with index k. We see that the initial 3D implicit problem becomes a set of L implicit 2D problems that can be solved independently. Finally, for a given spectral component (m, n) and a given mode with index k the implicit equation to be solved is:

$$\left[\mathbf{Q_B}^{-1}.\underline{\mathsf{d}}\right]_k = \left[(1 - \delta t^2 \left[\delta_\mathbf{B}\right]_k \mathbf{m}_*^2 \Delta')_n^m\right]^{-1} \left[\mathbf{Q_B}^{-1}.\underline{\mathsf{d}}^{\bullet\bullet\bullet}\right]_k \tag{5.38}$$

In the case where the linearized map factor $\mathbf{m}_*$ is a constant, the solution of the semi-implicit system therefore results in a scalar inversion $(1 - \delta t^2 \left[\delta_\mathbf{B}\right]_k \mathbf{m}_*^2 \Delta')_n^m$ for each spectral component (m, n) and for each eigenmode of $\mathbf{B}$.
In the case of a spatially-variable map-factor $\mathbf{m}_*$ (so-called option "`LSIDG`"), the operator $\mathbf{m}_*^2 \Delta'$ becomes non-diagonal, and has to be inverted matrically. This so-called `LSIDG` strategy is computationnally efficient only if the linear operator which represents the multiplication by the spatially-variable linearised map-factor $\mathbf{m}_*$ writes as a diagonal-banded matrix in the space of spectral components, and with a very limited number of nonzero diagonals.
Once the inversion step is performed, the unknown $\underline{\mathsf{d}}$ is recovered through a multiplication of the obtained result by $\mathbf{Q_B}$. This finishes the determination of the semi-implicit variable $\underline{\mathsf{d}}$.

5.10.3 Determination of variables at $t + \Delta t$

Once $\underline{\mathsf{d}}$ has been determined, the computation of the other prognostic variables at $t + \Delta t$ is straightforward. First, the horizontal divergence is obtained by:

$$\underline{D'} = \left(1 - \delta t^2 c_*^2 \Delta' \mathbf{m}_*^2\right)^{-1} \left[\underline{D^\bullet} + \delta t^2 \Delta' \left(-RT^* \mathbf{G}^* + c_*^2\right) \underline{\mathsf{d}}\right] \tag{5.39}$$

This matrix inversion can thus be solved scalarly for each column vector when $\mathbf{m}_*$ is a constant, but requires the inversion of a banded diagonal matrix in the case of a spatially-variable linearised map-factor $\mathbf{m}_*$. The other variables are directly obtained from the original system:

$$\begin{aligned}
\underline{\mathcal{P}} &= \underline{\widetilde{\mathcal{P}}} + \delta t \left[\left(\mathbf{S}^* - \frac{C_p}{C_v}\right) \mathbf{m}_*^2 \underline{D'} - \frac{C_p}{C_v} \underline{\mathsf{d}}\right] \\
\underline{T} &= \underline{\widetilde{T}} + \delta t \left(-\frac{RT^*}{C_v} \mathbf{m}_*^2 \underline{D'} - \frac{RT^*}{C_v} \underline{\mathsf{d}}\right) \\
\underline{\pi_s} &= \underline{\widetilde{\pi_s}} + \delta t \left(-\pi_s^* \mathbf{N}^* \mathbf{m}_*^2 \underline{D'}\right)
\end{aligned}$$

5.11 Time Filter (for 3-TL schemes)

A time filter is used to damp fast time oscillations of variables due to the decorelation of consecutive time levels in leap-frog (3-TL) schemes. The formulation is the same as in ARPEGE: The filter can be uncentered, as shown in the expression for filtered X variable (noted $\overline{X}$):

$$\overline{X}^t = \varepsilon_1 \overline{X}^{t-1} + (1 - \varepsilon_1 - \varepsilon_2) X^t + \varepsilon_2 X^{t+1}$$

This time filter is applied in grid point space, and computations are splitted into two steps. The first part of $\overline{X}^t$, involving $\overline{X}^{t-1}$ and X^t, is performed at the end of grid point calculations, after the computation of the explicit guess has been done using non filtered values X^t for the tendencies at time t. The second part is made at the beginning of the grid point calculations for the following time step, in such a way that the starting point for the leap frog scheme is the completely filtered value.

Chapter 6

Initialisation/Coupling

(08/05/1998)

6.1 Specification of NH variables from an hydrostatic state

The problem of specification of NH variables from an hydrostatic (H) state arises when trying to define initial/coupling files for a NH model with data from an H model output file. In our case the NH model is Aladin, but the H model can be ARPEGE (thus configuration E927) or Aladin (thus configuration EE927). Of course the H model output files contain no information about the NH variables, and some information must therefore be "invented" to be incorporated in the NH files.
Some preliminary works (Bubnova et al., 1995) have shown that the two extra variables introduced by the non-hydrostatism ($\hat{\mathcal{P}}$, $\hat{d}$) adapt themselves to the flow essentially in a local and fast way. The evolution of the flow is not crucially dependent on the value of these fields.
The principle retained here for the specification of the two new variables thus obey to the two following constraints:

- Conservation of the hydrostatic equilibrium initially present in the H state
- Conservation of the acoustic equilibrium initially present in the H state (in the absence of initial diabatic sources of course).

Therefore, all NH simulations basically start with a NH state but in hydrostatic equilibrium, and free of acoustic disturbances. The second constraint specifies that no acoustic energy is introduced in the initial state due to the specification of the two new variables by itself, but of course the acoustic energy potentially introduced by the presence of diabatic sources must be excluded when expressing the constraint. In fact, in the present state of the Aladin NH version this is a false problem since the diabatic sources are eliminated from the $\hat{\mathcal{P}}$ prognostic equation (see future chapter on introduction of diabatism). The two constraints thus have the following mathematical expression:

$$[p-\pi](t=0) \equiv 0 \tag{6.1}$$
$$\left[\frac{d(p-\pi)}{dt}\right]_{\text{adiab}}(t=0) \equiv 0 \tag{6.2}$$

which write:

$$\mathcal{P}(t=0) \equiv 0 \tag{6.3}$$

$$\left(\frac{d\mathcal{P}}{dt}\right)_{\text{adiab}}(t=0) \equiv 0 \tag{6.4}$$

The equation (2.57) for the adiabatic evolution of $\mathcal{P}$ then writes initially,

$$\left(\frac{d\mathcal{P}}{dt}\right)_{\text{adiab}}(t=0) = -(1+\mathcal{P})\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right)$$

and the second constraint becomes:

$$\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right) \equiv 0$$

Replacing with the expression of D_3 given by (2.62), the specification for the initial vertical velocity derivative finally becomes:

$$\left(-g\frac{\partial w}{\partial \eta}\right) = m\frac{RT}{\pi}\left(-\nabla\mathbf{V} - \frac{1}{m}\frac{\pi}{RT}\nabla\phi.\frac{\partial \mathbf{V}}{\partial \eta} - \frac{C_v}{C_p}\frac{\dot{\pi}}{\pi}\right) \tag{6.5}$$

This variable $-g(\partial w/\partial \eta)$ is diagnosed in the `SUBROUTINE POS.F` under the name `ZVDT0`

Chapter 7

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Chapter 8

Appendix A: Misleading terminology for π

(19/11/2003)

Here, we argue that the name "hydrostatic-pressure" chosen for π by Laprise,1992, is not very happy, because misleading in the general case. This is for at least three fundamental reasons. We think that the name "mass-based" coordinate is much more appropriate.

The use of the π coordinate is not necessarily restricted to meteorological or even planetary-atmosphere problems. As outlined in Laprise, 1992, and Wood and Staniforth, 2003, the numerical constant g which appears in (9.1) is introduced for simplicity reasons. But we could perfectly define π by:

$$\frac{\partial \pi}{\partial z} = -\frac{p}{RT} \tag{8.1}$$

In this case, it appears clearly that π has not necessarily any link with the gravity. One can imagine for example that we want to predict the atmosphere inside a parallelepipedic spacecraft without gravity. For the longest direction we could choose a length-based coordinate (x, y), but for the shortest direction, we could indifferently choose a length-based coordinate z or a mass-based coordinate π, as defined in (8.1).

Now, we imagine that the air inside our spacecraft is resting. Since we are in weightless conditions, then the pressure is uniform in the whole spacecraft, and since the air is resting, the pressure is equal to the static one. Hence the static pressure is uniform while π is of course not uniform. In this particular framework, it cannot therefore be said that π is the "value of the pressure if the flow is static".

Hence, we can say that the use of a mass-based coordinate is not restricted to planetary-atmosphere problems. For problems other than planetary-atmosphere type the π coordinate is not an hydrostatic-pressure coordinate, but a mass-coordinate.

In the previous example, it could be argued that Laprise, 1992, gave this name because is deliberately restricted his scope to meteorological problems (although this does not appear in the title, this restriction appears implicitely as soon as in the first sentence). Hence in the case of his paper, the word "hydrostatic-pressure" coordinate could be argued to be valid in spite of the above discussions. But even for the meteorological problem, the word is bad chosen.

Now we imagine that we want to represent a vertically-bounded atmosphere with a rigid lid at a fixed physical height (this type of representation is useful for meso-scale studies, hence it is wishable to be able to describe it). In this case the concept of hydrostatic-pressure itself is not well-defined, because the pressure at rest is not necessarily equal to the value of π defined by the Laprise paper. Yet the π coordinate defined by (9.1) can still perfectly be used, although it does not necessarily represent the pressure at rest divided by g.

Even for meteorological problems with vertically unbounded atmospheres, the name "hydrostatic-pressure coordinate" may be unappropriate. This is the case for deep-atmospheres, in which the area of a ground-unit-area column increases with height, due to the spherical geometry of the atmosphere. Wood and Staniforth, 2003, have shown that in this case, the coordinate π is a mass-coordinate and not an hydrostatic-pressure coordinate. In effect for deep-atmospheres, the hydrostatic-pressure is not equal to the weight of the atmospheric (diverging) column above a point, due to the upward buoyancy force exerted on the diverging surface of the column by the neighbouring columns.

The major problem of using this name is that it may give the feeling that the hydrostatic pressure is equal to π. This is not true in the general case, namely when the hydrostatic-pressure itself is not a well defined-concept, and this may lead to false interpretations, especially with respect to the specification of the top BC. Our recommendation should be rather to think π conceptually as a space coordinate which is related to the variations of mass, but nothing more. Thinking π in terms of an hydrostatic-pressure is not recommended.

• **Summary**

The term "hydrostatic-pressure coordinate" sometimes applied to the π coordinate is valid only for very restricted frameworks (vertically unbounded shallow-atmospheres for planetary problems), hence it should be used only very carefully, since using this name can lead to some misundertandings. The name "mass-based coordinate" should be preferred in general.

Chapter 9

Appendix B: Top and bottom boundary conditions in mass-coordinate models, in Aladin and Aladin-NH

(19/11/2003)

9.1 introduction

The physical meaning of the various constraints imposed at the boundaries of the Aladin and Aladin-NH systems is examined in this chapter.
The current versions of Aladin and Aladin-NH assume at the top of the domain that the boundary condition is characterized by $p_T = \pi_T = 0$. However, this is a limitation which could be wishable to be removed, e.g. for very small scale experiments such as bubble experiments, for which only a limited portion of the vertical atmosphere is modelled. In the current version , this type of experiment leads to a big jump in vertical resolution at the last level, or to the addition of many levels near the top in order to try to avoid this resolution jump.
In this note, we try to derive the shape of the top BC which should be applied to relax the current assumption $p_T = \pi_T = 0$ and we also try to give a physical interpretation of the resulting top BC.

9.2 Fundamental indeterminacy of π

We want to describe the evolution of a portion of the atmosphere with a mass-type coordinate π. Following Laprise, 1992, we define our mass coordinate by stipulating that the continuity equation must be purely diagnostic. However, defined in this way, the π coordinate is not fully-determined:

$$\frac{\partial \pi}{\partial z} = -g\frac{p}{RT} \tag{9.1}$$

It is seen from the above equation that this requirement of a diagnostic continuity equation is not enough to uniquely define π mathematically: π is only defined to the extent of a constant $\pi_0(x,y,t)$ for each column [denoted by (x,y)], and at each time t. The function $\pi_0(x,y,t)$ is in fact not totally arbitrary, it must be continuous and differentiable in order to be able to derive formulae for the coordinate change.

It should be **strongly outlined** that a further specification of $\pi_0(x,y,t)$ is **totally arbitrary** and free. Applying such a specification will have the result to allow a non-ambiguous definition for π but, will not restrict **in any aspect** the generality of the framework. The specification of π_0 must be viewed exactly as the specification of the origin of the coordinates in a more conventional system with a length-based coordinate: specifying where is located the origin of the coordinate never restrict the framework of a physical system.

Of course, when building a model, it is handful (and maybe necessary) to remove this indeterminacy, because we have to practically solve the equations, but as long as we do not try to solve practically the equations, the indeterminacy can be left in the system, by allowing $\pi_0(x, y, t)$ to be whatever possible. Here we keep the most general possiblility for π_0 without specifying it any further, because this helps to understand more clearly the role of the various conditions applied at boundaries, especially for the hydrostatic system.
In contrast to the fundamental indeterminacy of π, the true pressure p is determined uniquely in the atmosphere. In the following discussions, we assume that the atmosphere is an ideal gas, hence, p is known uniquely, if ρ and T are known. Experimentally, p is directly accessible through barometric measurements by comparing the real pressure to the pressure of the empty space, taken as a reference.

9.3 Domain boundaries, materiality, and mass conservation

9.3.1 specification of the domain boundaries

Our goal, while designing Aladin or Aladin-NH, is to describe the evolution of a fluid inside an "atmospheric" domain, i.e. which is convex in z for any column (x, y). Since π is a vertical coordinate, the domain will also be convex in π for any (x, y) (not shown). Thus, the domain can be defined by:

$$\pi \in [\pi_T(x, y, t), \pi_s(x, y, t)] \tag{9.2}$$

where $\pi_T(x, y, t)$ and $\pi_s(x, y, t)$ are two continuous and differentiable arbitrary functions for time being.

9.3.2 specification of the materiality of domain boundaries

For any vertical coordinate μ, and any flow, the fact that the flow is tangential to a given moving surface $\mu = \mu_s(x, y, t)$ writes:

$$\left[\frac{d}{dt}(\mu - \mu_s)\right]_{\mu=\mu_s} = 0 \tag{9.3}$$

where (d/dt) is the Lagrangian derivative. This condition is purely "kinematic" and simply stipulates that the surface μ_s is a trajectory surface for the velocity field of the considered flow, i.e. there is no velocity flow across the surface μ_s.
If the flow describes the evolution of a material medium attached to a mass density field ρ, this condition becomes a "material" boundary condition, and stipulates that there is no mass-flux across the surface μ_s.
From now on, we impose the upper and lower boundaries of the atmospheric domain (π_T, π_s) to be material. Hence, for parcels located at the upper or lower boundaries. The latter conditions write:

$$\dot{\pi}_{(\pi=\pi_s)} = \frac{\partial \pi_s}{\partial t} + V_s.\nabla\pi_s \tag{9.4}$$

$$\dot{\pi}_{(\pi=\pi_T)} = \frac{\partial \pi_T}{\partial t} + V_T.\nabla\pi_T \tag{9.5}$$

9.3.3 Evolution of the column-integrated mass

The continuity equation is exactly the same for the H and the NH systems. From now on, we assume that there are no mass sources, hence, the continuity equation writes:

$$\nabla V + \frac{\partial \dot{\pi}}{\partial \pi} = 0 \tag{9.6}$$

where $\dot{\pi}$ is the Lagrangian derivative of π. Integrating vertically through the whole domain we thus have:

$$\dot{\pi}_{(\pi=\pi_s)} - \dot{\pi}_{(\pi=\pi_T)} = -\int_{\pi_T}^{\pi_s} \nabla V d\pi \tag{9.7}$$

Using the materiality conditions (9.5)–(9.5), this yields:

$$\frac{\partial \pi_s}{\partial t} - \frac{\partial \pi_T}{\partial t} = -\int_{\pi_T}^{\pi_s} \nabla V d\pi - V_s.\nabla \pi_s + V_T.\nabla \pi_T \tag{9.8}$$

This equation describes the evolution of the column-integrated mass of the atmospheric medium inside the specified domain.

9.3.4 Mass conservation in the domain

We can check that this definition of the domain boundaries and of the materiality condition insure the conservation of the mass if the (x, y) domain represents a closed surface.

$$\begin{aligned}
\frac{d}{dt}\left\{\oint_\Sigma \left[\int_{\pi_s}^{\pi_T} \left(-\frac{1}{g}\right) d\pi\right] d\Sigma\right\} &= \frac{1}{g}\frac{d}{dt}\left[\oint_\Sigma (\pi_s - \pi_T) d\Sigma\right] \\
&= \frac{1}{g}\oint_\Sigma \frac{\partial}{\partial t}(\pi_s - \pi_T) d\Sigma \\
&= \frac{1}{g}\oint_\Sigma \left[-\int_{\pi_T}^{\pi_s} \nabla.V d\pi - V_s.\nabla\pi_s + V_T.\nabla\pi_T\right] d\Sigma \\
&= \frac{1}{g}\oint_\Sigma \left[\nabla \int_{\pi_T}^{\pi_s} V d\pi\right] d\Sigma
\end{aligned}$$

$\Rightarrow$

$$\frac{d}{dt}\left\{\oint_\Sigma \left[\int_{\pi_T}^{\pi_s} \left(-\frac{1}{g}\right) d\pi\right] d\Sigma\right\} = 0 \tag{9.9}$$

which proves the mass conservation. For the previous derivation, the following mathematical identities have been used:

$$\frac{d}{dt}\left[\oint_\Sigma f(x, y, t) d\Sigma\right] = \oint_\Sigma \left(\frac{\partial f}{\partial t}\right)_{(x,y)} d\Sigma \tag{9.10}$$

$$\nabla \int_{\pi_T}^{\pi_s} g(x, y, \pi, t) d\pi = \int_{\pi_T}^{\pi_s} \nabla g \; d\pi + g(\pi_s).\nabla\pi_s - g(\pi_T).\nabla\pi_T \tag{9.11}$$

9.4 Top and bottom BCs for Hydrostatic Models

In this section, in order to better illustrate the analogy with NH models, we keep the π coordinate undetermined, that is, the origin $\pi_0(x, y, t)$ of the coordinate is not specified (except in subsection 9.5).

The value of p is uniquely defined from the ideal gas law for instance, but the local value of the coordinate π is assumed undetermined. However, the system is hydrostatic, hence for each column (x, y) and each time t,

$[\pi - p(x, y, \pi, t)]$ is a pure constant that we can choose to define as precisely being $\pi_0(x, y, t)$. Doing this we simply stipulate that the arbitrary function $\pi_0(x, y, t)$ will acts as a link between p and π:

$$p(x, y, \pi, t) = \pi - \pi_0(x, y, t) \tag{9.12}$$

It should be repeated that the more specific writing in (9.12) does not induce any loss of generality: as long as π_0 is not chosen explicitly, the full indeterminacy of π remains entirely.
Inserting (9.12), the hydrostatic equation writes:

$$\frac{\partial \phi}{\partial \pi} = -\frac{RT}{\pi - \pi_0} \tag{9.13}$$

Hence:

$$\phi_T - \phi_s = \int_{\pi_T}^{\pi_s} \frac{RT(x, y, \pi, t)}{\pi - \pi_0(x, y, t)} d\pi \tag{9.14}$$

We are now in position to conceptually introduce physical top and bottom BCs. The bottom BC is imposed as "rigid" while the top boundary can be chosen as "rigid" or "elastic".

Rigid bottom BCs

For the bottom BC, we restrict ourselves to a rigid BBC, as traditionally in meteorology:
• $\phi_s = \Phi_s(x, y)$
where $\Phi_s(x, y)$ is an explicitly specified function of the horizontal (the earth's orography).

Rigid top BCs

The choice of a rigid top BC is not very natural for models formulated in mass-type coordinate, but it could perfectly be retained. In this case, the top BC would write:

• $\phi_T = \Phi_T(x, y)$

where $\Phi_T(x, y)$ is an explicitely specified function of the horizontal (usually a pure constant in practice). In this case, π_s and π_T will be functions of both (x, y) and t. They will be determined by solving the system formed by (9.8), (9.14), after an explicit specification of π_0. This leads to solving an integral implicit equation, and this is why the specification of a rigid top BC is not very natural for a mass-based coordinate model.

Elastic top BCs

The elastic top BC writes:

• $p_T = p_T(x, y)$

where $p_T(x, y)$ is an explicitely specified function of the horizontal (usually a pure constant). In this case, we have:

$$\pi_T(x, y, t) = p_T(x, y) + \pi_0(x, y, t), \tag{9.15}$$

thus, the explicit specification of π_0 will immediately lead to the explicit specification of π_T. Then π_s is determined through (9.8) and ϕ_T through (9.14). It should be noted that this system is much easier to solve than for the case of a rigid upper BC.

9.5 Practical applications for Hydrostatic Models

Now that the mathematical and physical meaning of the different constraints used for defining the BCs have been explained in the previous section, the last task is to remove the indeterminacy ot the π coordinate, in order to be able to solve the system practically. In this section, we examine some possible applications for the specification of the top BCs and the coordinate specification for hydrostatic models in the mass coordinate π.

9.5.1 Elastic top BCs

Elastic top with $\pi_T = p_T$

A very natural and simple choice is to relax the indeterminacy of π through (9.12) with the following explicit specification:

$$\pi_0(x, y, t) = 0 \tag{9.16}$$

This makes the definition of π unique. In this case (and only in this case) we have:

$$p(x, y, \pi, t) \equiv \pi, \tag{9.17}$$

For a practical application, the most natural choice for the elastic top BC is:

$$p_T(x, y, t) = \text{Const.} = p_{00} \tag{9.18}$$

Hence, by combining the elastic top BC and the coordinate origin specification, we can write :

$$\pi_T(x, y, t) = p_{00}. \tag{9.19}$$

The possibility to express directly the elastic top BC in terms of a constraint on π_T is thus linked to the particular fact that we have $p_T = \pi_T$ for this particular choice of the origin of the coordinate π.
As mentionned above, π_s is then determined through (9.8) and ϕ_T through (9.14).

Elastic top BC with $\pi_T = 0$

Another possibility to relax the indeterminacy of π is **not** to use (9.12), but to specify the condition:

$$\pi_T(x, y, t) = 0 \tag{9.20}$$

Note: One more time, this choice has no loss of physical generality with respect to the apparently wider choice $\pi_T(x, y, t) = \text{Const.}$ In particular, this perfectly allows the description of a vertically bounded atmosphere.

The elastic top BC consists in specifying p_T. Here also, the most natural choice for a practical application is:

$$p_T(x, y, t) = \text{Const.} = p_{00} \tag{9.21}$$

In this case, using (9.12), π_0 is given by:

$$\pi_0(x, y, t) = -p_{00} \tag{9.22}$$

It should be noted that in this case the geopotential is given by:

$$\phi - \Phi_s = \int_{\pi}^{\pi_s} \frac{RT(x, y, \pi, t)}{\pi + p_{00}} d\pi \tag{9.23}$$

As in the previous case, π_s is then determined through (9.8) and ϕ_T through:

$$\phi_T - \Phi_s = \int_0^{\pi_s} \frac{RT(x,y,\pi,t)}{\pi + p_{00}} d\pi \tag{9.24}$$

We see that if π_{00} is non-zero, the geopotential at top is bounded, thus allowing the description of a bounded atmosphere even if $\pi_T = 0$.

9.5.2 Rigid top BCs

Rigid top BC with $\pi_T = p_T$

We return to the choice of π through (9.12) with the following explicit specification:

$$\pi_0(x,y,t) = 0 \tag{9.25}$$

This makes the definition of π unique. In this case we have:

$$p(x,y,\pi,t) \equiv \pi \tag{9.26}$$

For a practical application, the most natural choice for a rigid top BC is:

$$\phi_T(x,y,t) = \mathrm{Const.} = \Phi_{00} \tag{9.27}$$

The two unknowns π_T and π_s are then determined by solving the system (9.8), (9.14). However, this system is not easy to solve.

Rigid top BC with $\pi_T = 0$

Here, we return to the possibility to relax the indeterminacy of π by the condition (without loss of physical generality):

$$\pi_T(x,y,t) = 0 \tag{9.28}$$

The rigid top BC is specified through:

$$\phi_T(x,y,t) = \mathrm{Const.} = \Phi_{00} \tag{9.29}$$

In this case, π_s is determined through:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^{\pi_s} \nabla V d\pi - V_s.\nabla \pi_s \tag{9.30}$$

The above specification of $\pi_T = 0$ implicitly determines π_0, through:

$$\pi_0(x,y,t) = -p_T(x,y,t), \tag{9.31}$$

and p_T then becomes an unknown of the system. The unknown p_T is then determined by solving:

$$\Phi_{00} - \Phi_s = \int_0^{\pi_s} \frac{RT(x,y,\pi,t)}{\pi + p_T} d\pi \tag{9.32}$$

This equation is not easy to solve, but indeed easier than the system obtained in the previous subsection 9.5.2.

9.5.3 Particular case of unbounded atmosphere

In the four approaches examined above, the particular case of an unbounded atmosphere is given by:

$$\pi_T \equiv 0 \tag{9.33}$$

$$p_t \equiv 0 \tag{9.34}$$

$$\pi_0 \equiv 0 \tag{9.35}$$

$$\Phi_{00} \equiv \infty \tag{9.36}$$

It should be also noticed that the unbounded atmosphere may be obtained as the limit case of both rigid and elastic top BC. In this sense, the unbounded atmosphere is both rigid and elastic. All these discussions become more or less trivial for an unbounded atmosphere.

9.5.4 Comments

The discussions in the four subsection above demonstrate that the elastic BC involves a constraint on the pressure field, while the unique determination of the coordinate involves a constraint on the π field. Imposing a constraint on the π field has fundamentally nothing to deal with an elastic condition but is just a necessary step to remove the indeterminacy of the coordinate. This will remain true for NH models.
On the other hand, imposing a rigid top BC is possible in the hydrostatic Aladin, and this can be legitimately done even if a constraint $\pi_T = \mathrm{const}$ is imposed. In this case, the true pressure at the top $p_T(x, y, t)$ becomes an unknown of the system.

9.6 Top and bottom BCs for NH Models

For a nonhydrostatic model, there is no longer any possible direct relation between p, π and π_0 as in (9.12) mainly because $(p - \pi)$ is no longer a "two dimensional" field (i.e. depending only on the column and the time). The full-determination of π_0, instead of being introduced via a fully-determined link between p, π and π_0 will in practice have necessarily to take another form.
Since the continuity equation is unchanged, we still have (9.8). However, (9.14) has to be replaced by:

$$\phi_T - \phi_s = \int_{\pi_T}^{\pi_s} \frac{RT}{p} d\pi \tag{9.37}$$

We are now in position to conceptually introduce physical top and bottom BCs. As for the hydrostatic system, the bottom BC is imposed as "rigid" while the top boundary can be chosen as "rigid" or "elastic".

Rigid bottom BCs

For the bottom BC, we restrict ourselves to a rigid BBC, as traditionally in meteorology:

$$\phi_s = \Phi_s(x, y) \tag{9.38}$$

where $\Phi_s(x, y)$ is an explicitly specified function of the horizontal (the model orography). Compared to the hydrostatic case, this constraint has an additional consequence, since it imposes also a constraint on the vertical acceleration at the bottom.

$$\dot{w}_s = V_s.[\nabla(V_s.\nabla\Phi_s)] \tag{9.39}$$

with the following consequence on the pressure field at the bottom:

$$\left[\frac{\partial \pi \mathcal{P}}{\partial \pi}\right]_s = -V_s.[\nabla(V_s.\nabla\Phi_s)] \tag{9.40}$$

The rigid BBC is used under this latter form in Aldin-NH, for the vertical momentum equation.

Rigid top BCs

The choice of a rigid top BC is not very natural for models formulated in mass-type coordinate, but it could perfectly be retained. In this case, the top BC would write:

- $\phi_T = \Phi_T(x, y)$

where $\Phi_T(x, y)$ is an explicitly specified function of the horizontal (usually a pure constant). In this case, π_s and π_T will be functions of both space (x, y) and time t. Similarly to the hydrostatic case, they will have to be determined by solving the system formed by (9.8), (9.37), after an explicit specification of π_0. Here also, solving this system is difficult and not very natural. A relationship similar to (9.40) would have to be applied at top for the vertical momentum equation.

Elastic top BCs

The elastic top BC writes:

- $p_T = p_T(x, y)$

where $p_T(x, y)$ is an explicitely specified function of the horizontal (usually a pure constant). This can be rewritten in terms of $\mathcal{P}$:

- $\mathcal{P}_T(x, y, t) = \dfrac{p_T(x, y)}{\pi_T(x, y, t)} - 1$

The fields $\pi_s(x, y, t)$ and $\pi_T(x, y, t)$ are obtained by solving the system formed by (9.8) altogether with the explicit specification of π_0.
The top geopotential ϕ_T is then given by (9.14).

9.7 Practical applications for NH models

Now that the mathematical and physical meaning of the different constraints used for defining the BCs have been explained, the last task is to remove the indeterminacy ot the π coordinate, in order to be able to solve the system practically. In this section, we examine some possible applications for the specification of the top BCs and the coordinate specification for non-hydrostatic models in the mass coordinate π.

9.7.1 Elastic top Bcs

Elastic top with $\pi_T = p_T$

For a practical application, the most natural choice for the elastic top BC is:

$$p_T(x, y, t) = \text{Const.} = p_{00} \tag{9.41}$$

A very natural and simple choice is then to specify $\pi_0(x, y, t)$ in such a way that:

$$\pi_T(x, y, t) = p_T = p_{00} \tag{9.42}$$

This makes the definition of π unique. Then, π_s is determined through (9.8) and ϕ_T through (9.14).

Elastic top with $\pi_T = 0$

The elastic top BC is still specified by:

$$p_T(x, y, t) = \text{Const.} = p_{00} \tag{9.43}$$

Another natural and simple choice is then to specify $\pi_0(x, y, t)$ in such a way that:

$$\pi_T(x, y, t) = 0 \tag{9.44}$$

This makes the definition of π unique. Then, π_s is determined through (9.8) and ϕ_T through (9.14).
Note: This approach has no loss of physically generality with respect to the one examined in the previous sub-section.

9.7.2 Rigid top BCs

Rigid top with $\pi_T = \pi_{00}$

For a practical application, the most natural choice for a rigid top BC is:

$$\phi_T(x, y, t) = \text{Const.} = \Phi_{00} \tag{9.45}$$

For determining the coordinate π, a simple solution is to choose π_0 in such a way that:

$$\pi_T(x, y, t) = \pi_{00} \tag{9.46}$$

This makes the definition of π unique. Then, π_s is determined through (9.8) and ϕ_T through (9.14).

Rigid top with $\pi_T = 0$

For a practical application, the most natural choice for a rigid top BC is:

$$\phi_T(x, y, t) = \text{Const.} = \Phi_{00} \tag{9.47}$$

For determining the coordinate π, another simple solution is to choose π_0 in such a way that:

$$\pi_T(x, y, t) = pi_0 0 \tag{9.48}$$

This makes the definition of π unique. Then, π_s is determined through (9.8) and ϕ_T through (9.14).

9.7.3 Comments

N.B: The constraint (9.42) does not mean physically in any way that the pressure is hydrostatic at the top! It simply stipulate that the origin of the coordinate is located in such a way that for the surface located at π_T, the pressure is given by p_T.
Another consequence is that it seems perfectly legal to use coordinate specification such as $\pi_T(x, y, t) = \text{const}$, altogether with a rigid top BC ($z_T = \text{const}$). This combination does not mean in any sense that we are mixing a rigid BC with an elastic BC, because $\pi_T(x, y, t) = \text{const}$ simply represent the definition of the coordinate origins, and has no physical meaning.

9.8 Case of Aladin-NH

In Aladin-NH an elastic top BC is chosen. In the current version, the restrictive hypothesis of an unbounded atmosphere ($p_T = 0$) is made. The implications of this hypothesis in the vertically-discretized model are first drawn. Then the possibility to extend the framework of Aladin-NH to a bounded atmosphere (still with an elastic top BC) is explored.

9.8.1 Current version of Aladin-NH

In the current version, the hypothesis of a vertically unbounded atmosphere is made. This implies:

$$p_T = 0 \tag{9.49}$$

However, we also have:

$$\pi_T = 0 \tag{9.50}$$

This latter condition makes that π is equal to the hydrostatic pressure (since the magnitude g of the gravitation vector is assumed uniform).
Mathematically speaking, $\mathcal{P}$ is indetermined at top, since $(p - \pi)$ and π both vanish at top. However, $\pi\mathcal{P} = (p - \pi)$ is perfectly determined at top.
The top BC of Aladin-NH is introduced by specifying the shape of the $\mathbf{L}(\mathcal{P})$ operator at the first level, that is $[\mathbf{L}(\mathcal{P})]_1$, in the vertical momentum equation. The current formulation is given by:
Hence the top BC writes:

$$[\mathbf{L}(\mathcal{P})]_1 = \frac{1}{\delta_1}\left[\frac{\pi_2\mathcal{P}_2 - \pi_1\mathcal{P}_1}{\pi_2 - \pi_1} - \frac{\pi_1\mathcal{P}_1 - (\pi\mathcal{P})_{\tilde{0}}}{\pi_1 - \pi_{\tilde{0}}}\right] \tag{9.51}$$

$\Rightarrow$

$$[\mathbf{L}(\mathcal{P})]_1 = \frac{1}{\delta_1}\left[\frac{\pi_2(\mathcal{P}_2 - \mathcal{P}_1)}{\pi_2 - \pi_1}\right] \tag{9.52}$$

These forms are the ones which are used in aladin-NH, as explained in the chapter on vertical discretisation.

9.8.2 Vertically bounded version of Aladin-NH

Here we explore the possibility to extend the current vertically unbounded version of Aldin-NH to vertically-bounded atmospheres. In this case we choose the simplest elastic top BC:

$$p_T = p_{00} \tag{9.53}$$

where p_{00} is a pure positive constant. Choosing the origin of the π coordinate (π_0) so as:

$$\pi_T = p_{00}, \tag{9.54}$$

then we have at top:

$$p_T - \pi_T = 0. \tag{9.55}$$

Hence:

$$(\pi\mathcal{P})_T = 0, \tag{9.56}$$

and the current top BC is thus still valid: the current form of the Laplacian operator at top is still valid since it just makes use of the value of $(\pi\mathcal{P})$ at top, and this value is unchanged.

9.8.3 Summary

Finally, the current vertically unbounded version of Aladin-NH can be readily extended to a vertically-bounded version with an elastic top BC at the pressure p_{00}, by simply specifying the values of the hybrid coordinate at top as being:

$$B_T = 0 \tag{9.57}$$

$$A_T = p_{00} \tag{9.58}$$

Chapter 10

Appendix C: Impact of the Additional Tridiagonal Matrix $\mathbf{T}^*$

(19/11/2003)

WARNING: The text of this chapter is provisional

10.1 Introduction

It has been seen in Chapter 4 that the major difference between the time-continuous space-continuous (3.34) and the time-continuous space-discretized (4.7) structure equations is the appearance of the $\mathbf{T}^*$ operator in the LHS of (4.7). The practical impact of this formal discrepancy between the space-continuous and the space-discretised evolutions is examined here.

Two approaches will be used to examine this impact:
- analytic approach in simplified case
- matrical approach in practical cases.

10.2 Analytic approach

General discussion about the impact of $\mathbf{T}^* \neq \mathbf{I}$ is rather complicated, and it is quite impossible to draw any conclusion in the general case. However, some insight can be gained by examining analytically the impact of this change in simplified frameworks.

The following hypotheses will be progressively applied in the remaining of this section:

- The mixed character of fast modes (gravity/elastic) is ignored and fast modes are considered as either pure-gravity of pure-elastic modes.
- The exceptions of the matrix elements at the top and bottom boundaries of the domain are ignored (this equivalent to examining the behaviour of a vertically-unbounded system).
- The vertical resolution (δz_l) is arbitrarily thin, thus we have $\delta_l = \delta z_l / H$ where H is the characteristic height of the atmosphere, assumed constant.
- The vertical resolution (δz) is regular. Hence $\delta_l = \delta = \delta z / H$

The time-continuous space-discretised structure equation (4.7) already contains the modification brought by the fact that $\mathbf{T}^* \neq \mathbf{I}$. The wave propagation is thus affected even in the time-continuous context. The analytic approach in this section first assumes a time-continuous context, then the impact on explicit and semi-implicit integrations is examined.

10.2.1 pure waves hypothesis

Starting from the time-continuous space-discretised structure equation (4.7) the hypothesis of pure modes allows to know which fast modes are most likely to be affected by the modification $\mathbf{T}^* \neq \mathbf{I}$. Hence we assume here that elastic and gravity modes can be considered as pure modes instead of mixed modes. The structure equation for the pure elastic modes writes:

$$\left[-\frac{1}{c_*^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\mathbf{m}_*^2\Delta' + \frac{\mathbf{L}^*}{rH_*^2}\right)\right] \mathsf{d} = 0 \tag{10.1}$$

Hence, the pure elastic waves will not be affected by the approximation since the propagation equation for these modes does not involve the last term of the complete structure equation.
On the other hand, the pure gravity waves propagation is likely to be affected. For gravity modes, the space-discretised structure equation writes:

$$\left[\frac{\partial^2}{\partial t^2}\left(\mathbf{m}_*^2\Delta' + \frac{\mathbf{L}^*}{rH_*^2}\right) + \frac{N_*^2}{r}\mathbf{T}^*\mathbf{m}_*^2\Delta'\right] \mathsf{d} = 0 \tag{10.2}$$

Hence the propagation of pure gravity modes is likely to be affected by the modification. The gravity-modes are hence the only fast modes that are likely to be affected by the modification.

In addition, it should be noted that the second constraint (C2) in equation (4.2) is involved only when one tries to perform the elimination of variables in order to determine the propagative properties of the allowed "fast" modes. The propagation of other "slow" modes (Rossby modes, "Lagrangian" perturbations embeded in the flow...) does not directly involve the fact that (C2) is fulfilled or not. In this point of view, the propagative properties of the slow-wave or non-wave processes is not likely ro be affected by the modification $\mathbf{T}^* \neq \mathbf{I}$.

10.2.2 Unbounded atmosphere and high resolution hypotheses

The unbounded atmosphere and high resolution hypotheses allow to determine which are the gravity-modes that are likely to be the most affected by the modification $\mathbf{T}^* \neq \mathbf{I}$.
The diagonal term of the $\mathbf{Q}^*$ matrix can be rewritten as:

$$(\delta_l^* - 2\alpha_l^*) = \frac{\left(\sqrt{\pi_{\tilde{l}}^*} - \sqrt{\pi_{\tilde{l}-1}^*}\right)^2}{\sqrt{\pi_{\tilde{l}-1}^*\pi_{\tilde{l}}^*}} = \frac{\delta\pi_l^{*2}}{\sqrt{\pi_{\tilde{l}-1}^*\pi_{\tilde{l}}^*}\left(\sqrt{\pi_{\tilde{l}}^*} + \sqrt{\pi_{\tilde{l}-1}^*}\right)^2} = \frac{\delta_l^{*2}}{4}\frac{2}{1+\sqrt{1+(\delta_l^{*2}/4)}}$$

We see that all elements of the diagonal matrix $\mathbf{Q}^*$ are positive (or zero for $\mathbf{Q}^*_{(1,1)}$ but this is ignored due to the vertically unbounded atmosphere hypothesis) and smaller than $\delta_l^{*2}/4$. When the resolution becomes thin enough, the latter expression tends (at the second-order) toward its continuous counterpart :

$$(\delta_l^* - 2\alpha_l^*) \simeq \delta_l^{*2}/4$$

Similarly, at the limit of high vertical resolutions, the internal expression of the $\mathbf{L}^*$ matrix becomes:

$$A_l^* = C_l^* = -B_l^*/2 = 1/\delta_l^{*2}$$

when the resolution increases, we can write for the matrix $\mathbf{T}^*$:

$$\mathbf{T}^* \quad \longrightarrow \quad \mathbf{I} + \frac{\delta_l^2}{4}\mathbf{L}^* \tag{10.3}$$

Similarly we can write:

$$\mathbf{T}^* \quad \longrightarrow \quad \mathbf{I} + \frac{1}{4}\mathbf{D}_2^* \tag{10.4}$$

where $\mathbf{D}_2^*$ is the canonical second order finite-difference (tridiagonal and symmetric) operator whose internal row is (0 ... 0, 1, -2, 1, 0 ... 0).
Hence $\mathbf{T}^*$ is a 3-points average operator whose internal row is (0 ... 0, 0.25, 0.5, 0.25, 0 ... 0). It can be seen from (10.3) that for a given vertical function, when the vertical resolution increases arbitrarily, the effect of the $\mathbf{T}^*$ matrix tends toward the effect of the identity operator. The error $(\mathbf{T}^* - \mathbf{I})$ is of the second-order in δ. This property is usually expressed by: "the $\mathbf{T}^*$ operator converges towards the identity operator with second-order accuracy in space". Hence the appearance of the $\mathbf{T}^*$ operator in the structure equation does not violate the part of general program of Aladin-NH which aims to maintain at least a second-order overall accuracy in time and space (see section 1.1).
However, determining the modification of the spatial structure and the frequency of normal modes is not trivial since the eigenmodes of $\mathbf{T}^*$ and $\mathbf{Q}^*$ in (10.2) are not necessarily identical. The problem consists in finding eigenvalues ω and eigenvectors $\underline{\mathsf{d}}$ such as the following equation is verified:

$$\left\{\left(\omega^2 - N_*^2\right)k^2 - \left[\frac{\mathbf{L}^*}{H_*^2}\left(\omega^2 + N_*^2 H_*^2 k^2 \mathbf{Q}^*\right)\right]\right\}\underline{\mathsf{d}} = 0$$

This problem cannot be analytically solved in the general case. The only way to come back to a classical (and analytically tractable) eigenmode problem is to assume a scalar matrix for $\mathbf{Q}^*$. This can be achieved by assuming a constant vertical resolution and still unbounded atmosphere.

10.2.3 Time-continuous propagation for constant vertical resolution

Assuming a constant vertical resolution allows to determine how the propagation of fast gravity modes will be modified by the $\mathbf{T}^*$ matrix compared to the exact case $\mathbf{T}^* = \mathbf{I}$.

If the vertical resolution is constant, then $H_*\delta_l = \delta z_l$ is a constant $H_*\delta = \delta z$ and the $\mathbf{Q}^*$ matrix is becomes a scalar matrix (in an unbounded atmosphere). Here we will assume that the vertical resolution is thin enough so that its diagonal values are $\delta^2/4 = (\delta z/2H_*)^2$. In this case, the determination of the normal mode of the modified system becomes a classical eigenmodes problem. The normal modes of the modified system (with $\mathbf{T}^* \neq \mathbf{I}$) have then the same vertical structure as for the exact system (with $\mathbf{T}^* = \mathbf{I}$). This vertical structure is in fact the one of the eigenvectors of the $\mathbf{L}^*$ operator. Let $(-n^2)$ be the eigenvalue associated with a given eigenmode of $\mathbf{L}^*/H_*^2$. The dispersion equation for this mode is thus:

$$\left(\omega^2 - N_*^2\right)k^2 + n^2\left(\omega^2 + N_*^2 H_*^2 k^2 \delta^2/4\right) = 0$$

Finally, for a given spatial structure $(k,\, n)$, the time-continuous frequency is:

$$\omega_c^2 = \frac{N_*^2 k^2}{k^2 + n^2}\left(1 - \frac{n^2 H_*^2 \delta^2}{4}\right) \tag{10.5}$$

where the subscript "c" outlines the fact that this frequency is for the time-continuous system.
We recognize the classical pure gravity wave time-continuous frequency equation, but modified by the factor in the RHS parenthesis, which appears due to the modification. This factor is close from 1 for long modes, but decreases for short modes. For the shortest $(2\delta z)$ mode, we have $\mathbf{D}_2^* = -4\mathbf{I}$, hence:

$$\mathbf{L}^* = \mathbf{D}_2^*/\delta^2 = -(4/\delta^2)\mathbf{I}$$

and:

$$n_{\max}^2 = \frac{4}{\delta^2 H_*^2}$$

Finally, for this shortest ($2\delta z$) mode, the factor in parentheses in (10.5) vanishes. The frequency (and also the phase and group velocities) of short gravity-modes is thus decreased in the case $\mathbf{T}^* \neq \mathbf{I}$ compared to the case $\mathbf{T}^* = \mathbf{I}$, especially for the shortest vertical modes, while the propagation of long modes is not significantly modified. However, since no imaginary part appears in the frequency ω_c, the modification will not generate amplitude errors (instability or damping).
At this point, the situation is thus as follows:

- Slow modes are not likely to be affected.
- Elastic modes are not likely to be affected
- Long gravity-modes are not likely to be affected.
- The only modes likely to be affected are the shortest gravity-modes.

10.2.4 Time-discretised explicit propagation for constant vertical resolution

The non-linear model is designed with formally the same vertical operators $\mathbf{L}$, $\mathbf{G}$, $\mathbf{S}$ and $\mathbf{N}$ as the linear ones, as seen in section 4.4. If we consider a stationnary state similar to the state used in the SI linear system, and we assume the model to be integrated explicitly, small perturbations will behave like in the previous time-continuous analysis, but with the eigenvalues of the non-linear operators. Hence, even in the complete model integrated explicitly, the short-gravity waves would have a distorted propagation.
Since the continuous phase speed is reduced, the stability criterion of the time scheme would hopefully not become more stringent for the explicit system than in the case $\mathbf{T}^* \neq \mathbf{I}$.

10.2.5 Time-discretised semi-implicit propagation for constant vertical resolution

In a similar way as in the time-continuous case, the vertical operators can be grouped in the Helmholtz equation of the time-discretised semi-implicit scheme:

$$\left[1 - \Delta t^2 c_*^2 \left(\Delta' + \frac{\mathbf{L}^*}{H_*^2}\left(1 + \Delta t^2 N_*^2 H_*^2 \Delta' \mathbf{Q}^*\right)\right) - \Delta t^4 N_*^2 c_*^2 \Delta'\right] \underline{\mathsf{d}}^+ = \underline{\mathsf{d}}^{\bullet\bullet} \tag{10.6}$$

As in the time-continuous case, the propagation of gravity-waves will only be affected. Some gravity-mode will moreover not be affected, provided that:

$$\Delta t^2 N_*^2 H_*^2 k^2 \mathbf{Q}^* \ll \mathbf{I}$$

Since $\mathbf{Q}^*$ is a diagonal matrix whose magnitude tends towards $\delta^2/4$ when the vertical resolution increases, its impact will therefore become negligible if $\Delta t N_* H_* k\,\delta/2 \ll 1$. Using $H_*\delta = \delta z$ for the physical depth of the layers, the condition writes:

$$k \ll \frac{2}{N_*\,\Delta t\,\delta z}$$

For the shortest horizontal modes the propagation of the gravity waves will be affected by the modification, while for longer horizontal modes, the propagation is not likely to be affected.

Since the semi-implicit system is still linear, the SI dispersion equation is still given by the general expression:

$$\omega_{\mathrm{SI}}\Delta t = \arctan\left(\omega_c \Delta t\right)$$

where ω_c is given by (10.5), and ω_{SI} is the frequency for the semi-implicit scheme. Therefore, the pure semi-implicit linear system still is unconditionally stable. For those modes which propagation is modified by the approximation, the modification still consists in decreasing the propagation velocities by a factor linked to the concerned eigenvalue of $\mathbf{T}^*$. Nevertheless, the modification appears to be of the same kind as the one brought by the SI treatment, i.e. it consists in adding an extra dispersivity to the natural continuous one. Finally, the expected impact of the modification is rather weak and should be overshaded by the impact of the SI scheme itself.

10.3 Matrical approach

In the particular cases examined above, the $\mathbf{Q}^*$ matrix is scalar and the propagation of the fast modes could be examined analytically because the spatial structure of the mode was not changed and only their frequency was modified. For a general discretisation, it is impossible to theoretically determine the propagation of the fast modes. However, for any given vertical discretisation, the propagation can be studied numerically.

Drawing from the previous section, the points which are important to be checked numerically are then:
- The positiveness of the eigenvalues of $\mathbf{T}^*$ (negative eigenvalues would lead to instability).
- The modification of the pure gravity-modes frequencies (in particular for the shortest ones).
- The modification of the vertical structure of the normal modes (in particular for the shortest ones).

In the following, these points are checked for the current operational L41 discretization of ARPEGE/Aladin. The eigenvalues of $\mathbf{T}^*$ for the operational L41 vertical discretisation are depicted in Fig. ??? sorted in decreasing order. They are all real and positive, hence there is no amplitude error (the scheme is not amplifying or damping). In fact, since the diagonal element of $\mathbf{Q}^*$ is smaller than $\delta_l^2/4$, it can be intuitively anticipated that the eigenvalues of $\mathbf{T}^*$ will always be positive. However, this has not been positively demonstrated so far.

gravity-modes frequencies (to be written)

vertical structure of the normal modes (to be written)

10.4 Summary

The impossibility to fulfil the second constraint leads to a modification of the propagation of the gravity waves. Altough the vertically-discretised system does not formally converge towards the continuous one as the vertical resolution increases, this can be considered as not important for the model for the following reasons:

- The stability of the model is not endangered

- The modification does not involve the slow part of the flow but only the fast modes.

- The large majority of the waves are not significantly affected

- For those waves which are affected, the modification is not dramatic, it only consists in an additional dispersivity probably overshaded by the intrinsic dispersivity of the semi-implicit scheme.

- The waves which play a meteorological role are not significantly affected by the modification.

- In any case, it should never be forgotten that the overall meteorological importance of fast wave is always assumed to remain weak in a NWP model

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