Scientific Documentation

for Aladin NH model

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Chapitre 1

Continuous Equations in the Cartesian System

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1.1 Introduction

The use of the "η"-coordinate system, introduced by Laprise, 1992, which is based upon the hydrostatic pressure, make possible the numerical solution of compressible equations, even in the case of flows over orography. With the non-hydrostatic coordinate systems classically used until now, which were based upon height or true pressure, the introduction of the orography led to an inextricable formulation of the compressible equations. For the anelastic system, the presence of orography results in an elliptic equation for the pressure containing spacially and temporally varying coefficients, making it very difficult to solve with precision.

In a first time, the system of continuous compressible equations in the cartesian geometry for the various plane projections allowed by ARPEGE/Aladin is introduced here. Then the linear version of the system is derived in order to show the validity of the structure equation for describing both gravity and acoustic waves present in the real atmosphere. The linear system is also used as a starting point to design the semi-implicit scheme.

1.2 Continuous Equations in Presence of Orography

The basic system in the presence of orography is derived here, in the cartesian geometry. This system will be used in the whole following.

1.2.1 The vertical coordinate "hydrostatic pressure", π

Before defining the hybride coordinate η, it is convenient to introduce the "pressure coordinate" π, which is used as a departure point. Without entering in the detail of the algebraic justification of it , let's just say that π is defined in each point by the weight of the unit area of the air column above the point. So the name "hydrostatic pressure" is well adapted to this coordinate.
The first remark which has to be done is that this coordinate (which is in fact more a massic coordinate than a strictly speaking true pressure coordinate) is, *by construction*, monotonic with respect to height. This is obviously not the case for the true pressure itself, for this local variable can decrease with height. This is

likely to occur in the case of "very agitated" flows only, which are certainly beyond the domain of numerical forecasting, but this first remark allows to pressent a domain of validity for this system extending far more than the restricted one of the meteorological numerical forecast.
To illustrate the sense of the previous remark in a intuitive way, it is convenient to imagine what would happen in the case of a rapid flow (let's say subsonic) in the area situated at the rear of a montaneous ridge. The local pressure is certainly dramatically weak in this small special area, due to the dynamical depressurisation, analogous to the one which occurs at the vicinity of a plane wing extrados. At the opposite, the hydrostatic pressure in the whole area will be determinated only by the weight of the atmosphere around the area, i. e. the repartition of temperature in the regions above the one considered here. Another suggestive way to illustrate the difference between static and dynamic pressure is to remark that on planes, manometers placed outboard can be used to measure the relative speed of the plane, and not the weigth of air above it.
The π coordinate is defined in the local geographical (x, y, z) cartesian coordinate system by:

$$\frac{\partial \pi}{\partial z} = -\rho g$$

that is:

$$\pi(x, y, z, t) = \int_z^{+\infty} \rho(x, y, z', t) g dz'$$

where Ox points toward east, Oy toward north, and Oz towards zenith.

1.2.2 Compressible system in π coordinate

The system describing the flow can be obtained from the classical height coordinate one by simply expressing the local geometry of the coordinate change:

$$\begin{aligned} \frac{\partial}{\partial z} &= -\frac{p}{RT} g \frac{\partial}{\partial \pi} \\ \nabla_z = \nabla_\pi &+ \frac{p}{RT} (\nabla_\pi \phi) \frac{\partial}{\partial \pi} \end{aligned}$$

where ϕ is the geopotential ($\phi = gz$).
The π-coordinate system then writes, in the local frame:

$$\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla p + \frac{\partial p}{\partial \pi}\nabla\phi = \mathcal{V} \tag{1.1}$$

$$\gamma\frac{dw}{dt} + g\left(1 - \frac{\partial p}{\partial \pi}\right) = \gamma\mathcal{W} \tag{1.2}$$

$$\nabla\mathbf{V} + \frac{\partial \dot{\pi}}{\partial \pi} = 0 \tag{1.3}$$

$$\frac{dT}{dt} - \frac{RT}{C_p}\frac{1}{p}\frac{dp}{dt} = \frac{Q}{C_p} \tag{1.4}$$

$$\frac{dp}{dt} + \frac{C_p}{C_v} p D_3 = \frac{Qp}{C_v T} \tag{1.5}$$

$$\frac{d\phi}{dt} = gw \tag{1.6}$$

$$\frac{\partial \phi}{\partial \pi} = -\frac{RT}{p} \tag{1.7}$$

with the following notations:

- $\mathbf{V}$: Horizontal wind vector.

- ∇ : Horizontal gradient on constant π surfaces.
- p : Local *true* pressure.
- w : Vertical velocity (dz/dt).
- D_3 : True 3-dimensional divergence of the wind.
- $\mathcal{V}, \mathcal{W}, Q$: Physical components of the forcing ($\mathcal{V}$ includes Coriolis term).
- γ : tracer $\gamma = 0$ (resp. 1) for degenerating the system to the hydrostatic one (resp. not). In this documentation γ is assumed to be 1.

1.2.3 The hybrid vertical coordinate η

The hybrid pressure coordinate η makes possible the representation of a bottom boundary condition in the presence of orography. This coordinate can be introduced from π in exactely the same way as the pressure hybrid coordinate is usually introduced from pressure in hydrostatic models:

$$\pi(x, y, \eta, t) = A(\eta) + B(\eta)\pi_s(x, y, t)$$

where π_s is the ground *hydrostatic* pressure (i.e. the weight of a unit area air column above the ground).
The variation domain of η can be defined by $[\eta_s, \eta_T]$. We impose without any loss of generality the two extremal values for η coordinate, *i.e*: $\eta_s = 1$ and $\eta_T = 0$ since the definition is an implicit one.

Lower boundary definition

The lower boundary definition is imposed by the fact that the bottom of the domain must follow the topography. For $\eta = 1$ we must then have $\pi = \pi_s$, which is equivalent to:

$$\begin{aligned} A(1) &= 0 \\ B(1) &= 1 \end{aligned}$$

Upper boundary definition

We restrict the model descriptions to domains which are delimited at the top by pure isobaric surfaces. This mean that for $\eta = 0$, the hydrostatic pressure π is a constant named π_T. This definition writes, in terms of A and B:

$$\begin{aligned} A(0) &= \pi_T \\ B(0) &= 0 \end{aligned}$$

In a first stage, we consider that $\pi_T = 0$, but since ARPEGE/Aladin is considered as a community tool, we plan to implement the possibility to run the model with a "ceiling" at pressure $\pi_T \neq 0$.

Kinematic boundary conditions

The kinematic boundary conditions express that the fluid flux is null through the limits of the domain:

$$\dot{\eta}(1) = \dot{\eta}(0) = 0$$

As in ARPEGE (case $\delta m = 1$), the conservation of the atmospheric air total mass during evaporation/precipitation at ground level can be handled trough a specification of $\dot{\eta}(1)$. This aspect is not discussed here.

NOTATION: We designate by $m(x,y,\eta,t)$ the quantity:

$$m = \frac{\partial \pi}{\partial \eta}$$

The transformation rules from π toward η are:

$$\begin{aligned} \frac{\partial}{\partial \pi} &= \frac{1}{m}\frac{\partial}{\partial \eta} && (1.8) \\ \nabla_\pi &= \nabla_\eta - (\nabla_\eta \pi)\frac{1}{m}\frac{\partial}{\partial \eta} && (1.9) \end{aligned}$$

Which yields:

$$\begin{aligned} m &= \frac{dA}{d\eta} + \pi_s \frac{dB}{d\eta} && (1.10) \\ \nabla_\eta \pi &= B\nabla_\eta \pi_s = B\pi_s \nabla(\ln \pi_s) && (1.11) \end{aligned}$$

1.2.4 Compressible system in hybrid coordinates η

The system cast in π coordinates can easily be transformed to η by application of the transformation rule between the two systems:

$$\begin{aligned} \frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla \phi &= \mathcal{V} && (1.12) \\ \gamma \frac{dw}{dt} + g\left(1 - \frac{1}{m}\frac{\partial p}{\partial \eta}\right) &= \gamma \mathcal{W} && (1.13) \\ \frac{dT}{dt} - \frac{RT}{C_p}.\frac{1}{p}\frac{dp}{dt} &= \frac{Q}{C_p} && (1.14) \\ \frac{dp}{dt} + \frac{C_p}{C_v}pD_3 &= \frac{Qp}{C_v T} && (1.15) \\ \frac{\partial m}{\partial t} + \nabla(m\mathbf{V}) + \frac{\partial}{\partial \eta}(m\dot{\eta}) &= 0 && (1.16) \\ \frac{d\phi}{dt} &= gw && (1.17) \\ \frac{\partial \phi}{\partial \eta} &= -m\frac{RT}{p} && (1.18) \end{aligned}$$

The expression for the true 3 dimensional divergence D_3 is:

$$D_3 = \nabla.\mathbf{V} + \frac{1}{m}\frac{p}{RT}\nabla\phi.\left(\frac{\partial \mathbf{V}}{\partial \eta}\right) - \frac{g}{m}\frac{p}{RT}\frac{\partial w}{\partial \eta}$$

The integration of the continuity equation on the vertical through the whole depth of the atmosphere leads the the surface hydrostatic pressure tendency equation:

$$\frac{\partial \pi_s}{\partial t} + \nabla \int_0^1 m\mathbf{V}d\eta = 0 \qquad (1.19)$$

In the same way, integrating from the top to the current level gives the pseudo-vertical velocity in η coordinate:

$$m\dot{\eta} = B\int_0^1 \nabla.m\mathbf{V}d\eta - \int_0^\eta \nabla.m\mathbf{V}d\eta$$

Finally the substantive derivative of hydrostatic pressure $\dot{\pi}$ (classically named ω) can be obtained from the two previous equations, and leads to the following diagnostic relation:

$$\omega = \dot{\pi} = \mathbf{V}.\nabla\pi - \int_0^\eta \nabla.m\mathbf{V}d\eta$$

1.2.5 Formulation with reduced non-hydrostatic pressure departure "$\hat{\mathcal{P}}$" and vertical divergence "$\hat{d}$"

Traditionnally, the problem of big cancelling terms in vertical velocity equation is avoided by using the non-hydrostatic pressure departure as an historical variable instead of true pressure. To insure weak vertical variations (a property which can be important for vertical advections), we further rescaled this departure by the basic state pressure:

$$\hat{\mathcal{P}} = \frac{p - \pi}{\pi_r(\eta)}$$

This rescaling also simplifies the set of linearized equations.
In a similar way, the solution of the semi-implicit scheme calls for a change towards a pseudo vertical divergence:

$$\hat{d} = -g \frac{\rho_r(\eta)}{m_r(\eta)} \frac{\partial w}{\partial \eta}$$

in replacement of the vertical velocity variable. Additionally this method will allow a natural staggering of vertical velocities and historical variables in the vertical discretization.
The quantities indexed by r are related to an homogeneous arbitrary reference state:

$\pi_r(\eta)$: pressure profile in the reference basic state.

$m_r(\eta) = (\partial \pi_r / \partial \eta)$.

$T_r(\eta)$: temperature in the reference state.

$\rho_r(\eta) = [\pi_r(\eta)/RT_r(\eta)]$.

Since the reference state is homogeneous,

$$\begin{aligned} \frac{d\rho_r}{dt} &= \dot{\eta}\frac{d\rho_r}{d\eta} \\ \frac{d(1/m_r)}{dt} &= -\frac{1}{m_r^2}\dot{\eta}\frac{dm_r}{d\eta} \end{aligned}$$

from the definition of $\hat{d}$, one obtains the following evolution equation for $\hat{d}$:

$$\frac{d(\hat{d})}{dt} = -g\left[-\frac{\rho_r}{m_r^2}\,\dot{\eta}\,\frac{dm_r}{d\eta}\frac{\partial w}{\partial \eta} + \frac{1}{m_r}\,\dot{\eta}\,\frac{d\rho_r}{d\eta}\frac{\partial w}{\partial \eta} + \frac{\rho_r}{m_r}\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right)\right]$$

one can write:

$$\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right) = \frac{\partial \dot{w}}{\partial \eta} - \frac{\partial \mathbf{V}}{\partial \eta}.\nabla w - \frac{\partial \dot{\eta}}{\partial \eta}.\frac{\partial w}{\partial \eta}$$

which yields:

$$\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right) = g\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial \pi_r \hat{\mathcal{P}}}{\partial \eta}\right) - \frac{\partial \mathbf{V}}{\partial \eta}.\nabla w - \frac{\partial \dot{\eta}}{\partial \eta}.\frac{\partial w}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial \eta}$$

finally, the prognostic equation for $\hat{d}$ writes:

$$\begin{aligned} \frac{d(\hat{d})}{dt} = &- g^2\frac{\rho_r}{m_r}\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial \pi_r \hat{\mathcal{P}}}{\partial \eta}\right) + g\frac{\rho_r}{m_r}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w + g\frac{\rho_r}{m_r}\frac{\partial \dot{\eta}}{\partial \eta}.\frac{\partial w}{\partial \eta} \\ &- g\frac{1}{m_r}\,\dot{\eta}\,\frac{\partial w}{\partial \eta}\frac{d\rho_r}{d\eta} + g\frac{\rho_r}{m_r^2}\,\dot{\eta}\,\frac{\partial w}{\partial \eta}\frac{dm_r}{d\eta} - g\frac{\rho_r}{m_r}\frac{\partial \mathcal{W}}{\partial \eta} \end{aligned} \tag{1.20}$$

Similarly, we derive the prognostic equation for $\hat{\mathcal{P}}$ from the equation of pressure:

$$\begin{aligned}
\frac{d\hat{\mathcal{P}}}{dt} &= \frac{1}{\pi_r}\left(\frac{dp}{dt}-\frac{d\pi}{dt}\right)-\frac{(p-\pi)}{\pi_r^2}\frac{d\pi_r}{dt} \\
&= -\frac{C_p}{C_v}\frac{p}{\pi_r}D_3+\frac{1}{\pi_r}\frac{Qp}{C_vT}-\frac{\dot{\pi}}{\pi_r}-\frac{(p-\pi)}{\pi_r^2}\dot{\eta}\frac{d\pi_r}{d\eta} \qquad (1.21)
\end{aligned}$$

Using the new variables and combining the temperature and pressure equation, the original η-system can thus be rewritten:

$$\begin{aligned}
\frac{d\mathbf{V}}{dt}+\frac{RT}{p}\nabla p+\frac{1}{m}\frac{\partial p}{\partial \eta}\nabla\phi &= \mathcal{V} \qquad (1.22)\\
\frac{d(\hat{d})}{dt}+g^2\frac{\rho_r}{m_r}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial \pi_r\hat{\mathcal{P}}}{\partial\eta}\right)-g\frac{\rho_r}{m_r}\frac{\partial\mathbf{V}}{\partial\eta}.\nabla w & \\
-g\frac{\rho_r}{m_r}\frac{\partial\dot{\eta}}{\partial\eta}.\frac{\partial w}{\partial\eta}+g\frac{1}{m_r}\dot{\eta}\frac{\partial w}{\partial\eta}\frac{d\rho_r}{d\eta}-g\frac{\rho_r}{m_r^2}\dot{\eta}\frac{\partial w}{\partial\eta}\frac{dm_r}{d\eta} &= -g\frac{\rho_r}{m_r}\frac{\partial\mathcal{W}}{\partial\eta} \qquad (1.23)\\
\frac{dT}{dt}+\frac{RT}{C_v}D_3 &= \frac{Q}{C_v} \qquad (1.24)\\
\frac{d\hat{\mathcal{P}}}{dt}+\frac{C_p}{C_v}\frac{p}{\pi_r}D_3+\frac{\dot{\pi}}{\pi_r}+\frac{(p-\pi)}{\pi_r^2}\dot{\eta}\frac{d\pi_r}{d\eta} &= \frac{1}{\pi_r}\frac{Qp}{C_vT} \qquad (1.25)\\
\frac{\partial\pi_s}{\partial t}+\nabla\int_0^1 m\mathbf{V}d\eta &= 0 \qquad (1.26)\\
\frac{d\phi}{dt} &= gw \qquad (1.27)\\
\frac{\partial\phi}{\partial\eta} &= -m\frac{RT}{\pi+\pi_r\hat{\mathcal{P}}} \qquad (1.28)
\end{aligned}$$

1.3 Derivation of the Linearised System

The propagation characteristics of small amplitude waves in the fluid can be deduced from the analysis of the linearised set of equations around a fixed basic state representative of the typical ambient conditions in the atmosphere. Once the waves allowed in the fluid are known, the linear terms involved in their propagation can be treated in a semi implicit way, in order to increase the maximum time step for linear stability.

In this section, the linearised system is derived, the basic state being chosen such as to allow both gravity and acoustic waves control.

1.3.1 Basic state

The chosen basic state is isothermal and resting, thus hydrostatically balanced. The orography is not represented in the basic state excepted by a uniform background value for the ground geopotential. The basic state orography is therefore different from the real one. Classically, the basic state quantities are represented by an asterisc.
We thus have:

$$\begin{aligned}
T^* &= Cte \\
u^* &= v^* = w^* = 0 \\
\phi_s^* &= \phi^*_{(\eta=1)} = Cte
\end{aligned}$$

The basic state is horizontally homogeneous (i.e. on the iso-η surfaces due to the homogeneity of orography). The variables ϕ^*, π^* and m^* are functions of η only. The equation of state for the basic state is:

$$\frac{d\phi^*}{d\eta} = -RT^* \frac{m^*}{\pi^*} \tag{1.29}$$

The vertical integration of the previous equation shows that the function $(\phi^* + RT^* \ln \pi^*)$ is independant of η:

$$\phi^* + RT^* \ln \pi^* = Cte$$

We set the value of this constant by specifying the hydrostatic pressure at ground in the basic state:

$$\pi^*_{(\eta=1)} = \pi^*_s (= Cte)$$

The true pressure in the basic state p^* is of course equal to its hydrostatic counterpart π^* since the basic state is under hydrostatic equilibrium: $p^* = \pi^*$, thus we have $\rho^* = (\pi^*/RT^*)$.

The basic state is now entirely defined. One can see that it is described by the choice of three arbitrary constants: T^*, ϕ^*_s, and π^*_s. This choice has some implication in terms of stability of the semi-implicit scheme (Simmons *et al.*, 1978).

The vertical profile of pressure in the basic state is a consequence of the choice of the A and B functions defining the vertical coordinate:

$$\pi^*_{(\eta)} = A_{(\eta)} + B_{(\eta)} \pi^*_s \tag{1.30}$$

$$m^*_{(\eta)} = \frac{dA}{d\eta} + \frac{dB}{d\eta} \pi^*_s \tag{1.31}$$

1.3.2 Definition of the deviation

For a given state of the atmosphere in the complete system, we define the deviation by the difference with the local value in the basic state. Classically, the quantities relatives to the deviation are primed. For the temperature and dynamical variables, we have thus:

$$\begin{aligned}
T &= T^* + T' \\
\phi &= \phi^*_{(\eta)} + \phi' \\
\phi_s &= \phi^*_s + \phi'_s \\
u &= u^* + u' = u' \\
v &= v^* + v' = v' \\
w &= w^* + w' = w' \\
D &= D^* + D' = D'
\end{aligned}$$

where $D = (\partial u/\partial x) + (\partial v/\partial y)$ represents the horizontal part of the wind divergence. Since $u^* = v^* = w^* = D^* = 0$, we use the original variables u, v, D in the linear model formulation.

If π and π_s represent the local values of hydrostatic pressure in the system, the deviation for current level and ground hydrostatic pressure are given by:

$$\begin{aligned}
\pi &= \pi^*_{(\eta)} + \pi' \\
\pi_s &= \pi^*_s + \pi'_s \\
m = &= m^* + m'
\end{aligned}$$

For the arbitrary reference state of the section (1.2.5), we choose the basic state (ϕ_s^*, T^*, π_s^*) of the linear sytem. In this way, the two variables $\hat{\mathcal{P}}$ and $\hat{d}$ are adimensionalized by the basic state background values:

$$\begin{aligned} \pi_r &= \pi^* \\ T_r &= T^* \end{aligned}$$

The definition for $\hat{\mathcal{P}}$ and $\hat{d}$ then revert to:

$$\hat{\mathcal{P}} = \frac{p - \pi}{\pi^*} \tag{1.32}$$

$$\hat{d} = -g\frac{\rho^*}{m^*}\frac{\partial w}{\partial \eta} \tag{1.33}$$

Since $p^* = \pi^*$ and $w_* = 0$, the basic state value of $\hat{\mathcal{P}}$ and $\hat{d}$ are zero, and these two variables are their own linearised counterpart. The adimensionalization of the two non-hydrostatic prognostic perturbation variables is necessary to keep reasonable magnitude for vertical advection terms in the linear model, a feature which has important implication in the stability of the semi-implicit scheme. In the presence of anabruptly varying vertical resolution, the vertical advection of these variables could not be easily stabilised if not normalised by a local background value.

$$\hat{\mathcal{P}}' = \hat{\mathcal{P}} \tag{1.34}$$

$$\hat{d}' = \hat{d} \tag{1.35}$$

1.3.3 General properties of the deviation

First we could note that ϕ_s' is time independant since ϕ_s^* and ϕ_s are time independant.
The variable π and its derivatives can be expressed in term of the ground values and A, B functions:

$$\begin{aligned} \pi^* &= A(\eta) + B(\eta)\pi_s^* \\ \pi' &= B\pi_s' \\ m' &= \frac{dB}{d\eta}\pi_s' \end{aligned}$$

Since $p = \pi + \pi' + \pi^*\hat{\mathcal{P}}$, the derivatives of p are:

$$\begin{aligned} \nabla p &= B\nabla\pi_s' + \pi^*\nabla\hat{\mathcal{P}} \\ \frac{\partial p}{\partial \eta} &= m + m^*\hat{\mathcal{P}} + \pi^*\frac{\partial\hat{\mathcal{P}}}{\partial \eta} \end{aligned}$$

1.3.4 Definition of the linearised system

Surface pressure tendency :

The linearisation of the the surface pressure tendency equation (1.19) gives:

$$\frac{\partial \pi_s'}{\partial t} + \nabla . \int_0^1 m^*\mathbf{V}d\eta = 0$$

that is:

$$\frac{\partial \pi_s'}{\partial t} + \int_0^1 m^* D d\eta = 0$$

Vertical divergence equation :

In the equation of the vertical divergence (1.20), the only source term giving a linear contribution is the first one, thus:

$$\gamma \frac{\partial \hat{d}}{\partial t} + g^2 \frac{\rho^*}{m^*} \frac{\partial}{\partial \eta} \left(\frac{1}{m^*} \frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \eta} \right) = 0$$

Thermodynamic equation :

The thermodynamic equation linearises to:

$$\frac{\partial T'}{\partial t} = -\frac{RT^*}{C_v} \left(D + \hat{d} \right)$$

since the linear form of D_3 is:

$$D_3 \to (D + \hat{d})$$

Omega equation

The linearised version of $\dot{\pi}$ (also noted traditionnally ω) is:

$$\omega' = B \frac{\partial \pi_s}{\partial t} + m^* \dot{\eta}$$

and the linearised version of the omega equation is:

$$\omega' = - \int_0^\eta m^* D d\eta$$

Pressure deviation equation :

The linearisation of the $\hat{\mathcal{P}}$ equation gives:

$$\frac{\partial \hat{\mathcal{P}}}{\partial t} = -\frac{C_p}{C_v} \left(D + \hat{d} \right) + \frac{1}{\pi^*} \int_0^\eta m^* D d\eta$$

State equation :

As $\pi^* (d\phi^*/d\eta) = -RT^* m^*$, the state equation can be linearised to:

$$\pi^* \frac{\partial \phi'}{\partial \eta} + \pi' \frac{d\phi^*}{d\eta} + \pi^* \hat{\mathcal{P}} \frac{d\phi^*}{d\eta} = -m' RT^* - m^* RT'$$

that is:

$$\frac{\partial \phi'}{\partial \eta} = m^* RT^* \frac{\pi'}{\pi^{*2}} + m^* RT^* \frac{\hat{\mathcal{P}}}{\pi^*} - m' \frac{RT^*}{\pi^*} - \frac{m^*}{\pi^*} RT'$$

The vertical integration gives the expression of the geopotential deviation:

$$\int_\eta^1 \frac{\partial \phi'}{\partial \eta} d\eta = -RT^* \left[\frac{\pi'}{\pi^*} \right]_\eta^1 - R \int_\eta^1 \frac{m^*}{\pi^*} T' d\eta + RT^* \int_\eta^1 \frac{\hat{\mathcal{P}}}{\pi^*} m^* d\eta$$

which takes the final form:

$$\phi' = \phi'_s - RT^* \frac{\pi'}{\pi^*} + RT^* \frac{\pi'_s}{\pi^*_s} + R \int_\eta^1 \frac{m^*}{\pi^*} T' d\eta - RT^* \int_\eta^1 \frac{\hat{\mathcal{P}}}{\pi^*} m^* d\eta$$

Divergence equation :

The vector linearised momentum equation writes:

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{RT^*}{\pi^*}\left(\nabla\pi' + \pi^*\nabla\hat{\mathcal{P}}\right) + \nabla\phi' = 0$$

taking the divergence of this equation leads to:

$$\frac{\partial D}{\partial t} = -\frac{RT^*}{\pi^*}\left(\Delta\pi' + \pi^*\Delta\hat{\mathcal{P}}\right) - \Delta\phi'$$

and finally, using the above geopotential expression:

$$\frac{\partial D}{\partial t} = -R\int_\eta^1 \frac{m^*}{\pi^*}\Delta T'\,d\eta + RT^*\int_\eta^1 \frac{m^*}{\pi^*}\Delta\hat{\mathcal{P}}d\eta - RT^*\Delta\hat{\mathcal{P}} - \frac{RT^*}{\pi_s^*}\Delta\pi_s' - \Delta\phi_s' \tag{1.36}$$

<u>REMARK</u>: In the hypothesis of infinitesimal meteorological perturbation $T' = u = v = w = D \simeq 0$, all the tendencies must remain infinitesimal. In our formulation of the linear system we have allowed an orography perturbation ϕ_s' around the basis state value ϕ_s^*, and once this perturbation ϕ_s' is specified, it cannot be assumed to tend towards zero as the other "meteorological" perturbations, and the divergence equation *apparently* produces tendencies which do not tend towards zero. However we show here that this is not the case since π_s' produces a compensating tendency.

When the meteorological perturbation tends towards zero, the surface pressure is given by:

$$\phi_s + RT^*\ln\pi_s = \phi_s^* + RT^*\ln\pi_s^*$$

or:

$$\pi_s' = -\frac{\pi_s^*}{RT^*}\phi_s'$$

thus exactely compensating the artificial tendency given by the geopotential term. In an infinitesimal perturbation case, the part of the two term related to the vertical profile variation of π^* compensate each other, which ensures the coherence of the linear system.

1.4 Derivation of the Structure Equation

1.4.1 Notations

We define here the basic state square of the acoustic phase speed, the characteristic height of the atmosphere, and the square of Brünt-Vaisalä frequency:

$$\begin{aligned} c^2 &= RT^*\frac{C_p}{C_v} \\ H &= \frac{RT^*}{g} \\ N^2 &= \frac{g^2}{C_pT^*} \end{aligned}$$

One also defines a set of vertical continuous linear operators by:

$$\begin{aligned} \partial^* X &= \frac{\pi^*}{m^*}\frac{\partial X}{\partial\eta} \\ \mathcal{G}^* X &= \int_\eta^1 \frac{m^*}{\pi^*}X\,d\eta \end{aligned}$$

$$
\begin{aligned}
\mathcal{S}^* X &= \frac{1}{\pi^*} \int_0^\eta m^* X \, d\eta \\
\mathcal{N}^* X &= \frac{1}{\pi_s^*} \int_0^1 m^* X \, d\eta \\
\mathcal{L}^* X &= \partial^* \left(\partial^* + 1 \right) X
\end{aligned}
$$

The following properties are true for these operators:

$$
\begin{aligned}
\partial^* . \mathcal{G}^* &= \mathcal{G}^* . \partial^* = -1 \\
\left(\partial^* + 1 \right) . \mathcal{S}^* &= \mathcal{S}^* . \left(\partial^* + 1 \right) = 1
\end{aligned}
$$

1.4.2 Elimination of T', π'_s, $\hat{\mathcal{P}}$

The evolution equations for perturbation in the linear system writes:

$$
\begin{aligned}
\frac{\partial D}{\partial t} &= -R\mathcal{G}^* \Delta T' + gH\mathcal{G}^* \Delta \hat{\mathcal{P}} - RT^* \Delta \hat{\mathcal{P}} - \frac{RT^*}{\pi_s^*} \Delta \pi'_s - \Delta \phi'_s && (1.37) \\
\frac{\partial T'}{\partial t} &= -\frac{RT^*}{C_v} (D + \hat{d}) && (1.38) \\
\frac{\partial \pi'_s}{\partial t} &= -\pi_s^* \mathcal{N}^* D && (1.39) \\
\frac{\partial \hat{\mathcal{P}}}{\partial t} &= \mathcal{S}^* D - \frac{C_p}{C_v} (D + \hat{d}) && (1.40) \\
\frac{\partial \hat{d}}{\partial t} &= -\frac{g}{H} \mathcal{L}^* \hat{\mathcal{P}} && (1.41)
\end{aligned}
$$

The $\hat{d}$ equation gives:

$$
\begin{aligned}
\frac{\partial^2 \hat{d}}{\partial t^2} &= -\frac{g}{H} \mathcal{L}^* \left[\mathcal{S}^* D - \frac{C_p}{C_v} (D + \hat{d}) \right] \\
\Rightarrow \quad \frac{\partial^2 \hat{d}}{\partial t^2} &= -\frac{g}{H} \mathcal{L}^* \mathcal{S}^* D + \frac{c^2}{H^2} \mathcal{L}^* \left(D + \hat{d} \right)
\end{aligned}
$$

which can be written:

$$
\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\mathcal{L}^*}{H^2} \right) \hat{d} = \frac{\mathcal{L}^*}{H^2} \left(-gH\mathcal{S}^* + c^2 \right) D \tag{1.42}
$$

The time derivative of the divergence equation gives:

$$
\frac{\partial^2 D}{\partial t^2} = -R\mathcal{G}^* \Delta \frac{\partial T'}{\partial t} + gH\mathcal{G}^* \Delta \frac{\partial \hat{\mathcal{P}}}{\partial t} - RT^* \Delta \frac{\partial \hat{\mathcal{P}}}{\partial t} - \frac{RT^*}{\pi_s^*} \Delta \frac{\partial \pi'_s}{\partial t}
$$

i.e:

$$
\frac{\partial^2 D}{\partial t^2} = c^2 \Delta (D + \hat{d}) - gH\mathcal{G}^* \Delta \hat{d} - gH\mathcal{A}_1^* \Delta D
$$

with:

$$
\mathcal{A}_1^* = \left(-\mathcal{G}^* \mathcal{S}^* + \mathcal{G}^* + \mathcal{S}^* - \mathcal{N}^* \right)
$$

It can easily be checked that this $\mathcal{A}_1^*$ operator is zero in this continuous approach. This constraint will also have to be fulfilled for the discrete operators case.

The divergence equation then become:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right) D = \left(-gH\mathcal{G}^* + c^2\right)\Delta\hat{d} \tag{1.43}$$

1.4.3 Structure equation

The elimination can be carried out by multiplying (1.42) by the left hand side term operator of (1.43), and by using the commutativity of temporal and horizontal derivation operators with vertical derivation operators:

$$\begin{aligned}
\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\mathcal{L}^*}{H^2}\right)\hat{d} &= \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\frac{\mathcal{L}^*}{H^2}\left(-gH\mathcal{S}^* + c^2\right) D \\
&= \frac{\mathcal{L}^*}{H^2}\left(-gH\mathcal{S}^* + c^2\right)\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right) D \\
&= \frac{\mathcal{L}^*}{H^2}\left(-gH\mathcal{S}^* + c^2\right)\left(-gH\mathcal{G}^* + c^2\right)\Delta\hat{d}
\end{aligned}$$

Note: we remark here that eliminating $\hat{d}$ instead of D would have led to an extra commutativity constraint on vertical operators. With this order of elimination, the group which has to be displaced to complete the substitution is a group involving only temporal and horizontal operators, thus commutating by nature with vertical ones. This is the reason for chosing this order, and it will be very important in the discrete case.
The latter equation develops in:

$$\frac{\partial^4}{\partial t^4} - c^2\frac{\partial^2}{\partial t^2}\left(\frac{\mathcal{L}^*}{H^2} + \Delta\right)\hat{d} - g^2\mathcal{L}^*\left(\mathcal{S}^*\mathcal{G}^* - \frac{C_p}{C_v}\mathcal{S}^* - \frac{C_p}{C_v}\mathcal{S}^*\right)\Delta\hat{d} = 0$$

An operator $g^2\mathcal{L}^*.\mathcal{A}_2^*$ applying to $\Delta\hat{d}$ appears. It is easy to show that this operator simplifies by virtue of the above mentionned properties of the continuous operators:

$$g^2\mathcal{L}^*.\mathcal{A}_2^*.\Delta\hat{d} = g^2\mathcal{L}^*\left(\mathcal{S}^*\mathcal{G}^* - \frac{C_p}{C_v}\mathcal{S}^* - \frac{C_p}{C_v}\mathcal{G}^*\right)\Delta\hat{d} = N^2c^2\Delta\hat{d}$$

Here too, this property of continuous operators will have to be taken into account when designing the discrete operators. We finally obtain the structure equation of the system. This equation describes the structure of the perturbation in the linear system in the 4D space. In particular, it allows to find the propagation of linear waves in the associated complete system.

$$\underline{\left[-\frac{1}{c^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\Delta + \frac{\mathcal{L}^*}{H^2}\right) + N^2\Delta\right]\hat{d} = 0} \tag{1.44}$$

1.4.4 Comments

First, it is fundamental to note that the obtained structure equation is formally identical to the one in the fully compressible z system, replacing ∂^* by: $-H(\partial/\partial z)$ (cf: Eckart, 1960). It is also identical to the one obtained in the fully compressible π system, replacing ∂^* by: $\pi(\partial/\partial\pi)$ (cf: Laprise, 1992).
As a consequence, the three systems behave exactly the same way for the waves they allow, and for their propagation characteristics. For example, all of them contain the possible propagation of the three rapid waves which are present in the real atmosphere: gravity waves, acoustic waves and Lamb waves.
Another important property of this equation is to be essentially a *local* one, only involving derivative operators. Unlike in the hydrostatic system, (cf: ARPEGE documentation e.g.) no integral operator is to be considered for knowing the propagation of waves.

Last, it should be noted that the coefficients of the operators are *constant* in time and on a η surface. The consequence is that for a spectral model, the solution of the structure equation reverts to the inversion of a set of one constant coefficient Helmholtz equation per level. The solution of this type of equation doesn't present any particular problem, and is analogous to the horizontal laplacian's inversion in the hydrostatic case.

1.5 Final Form of Dynamical Model Equations

1.5.1 Model variables

The prognostics variables of the model are:

- u, v: Horizontal wind.
- T: Temperature.
- $\hat{\mathcal{P}}$: Adimensionalised (by π^*) non-hydrostatic pressure.
- $\hat{d}$: pseudo vertical divergence.
- π_s: Surface hydrostatic pressure.
- q : water vapor specific content (in diabiatic version).

some other expressions are used as diagnostic variables:

- ϕ: Geopotential.
- D : Horizontal divergence.
- ζ: Vertical component of vorticity.
- D_3: 3 Dimensional divergence.
- $m\dot{\eta}$: Advective vertical velocity.
- $\mathcal{P}$: Adimensionalized (by π) non-hydrostatic pressure
- d: vertical divergence

1.5.2 Dynamical model Prognostic equations

The original system (1.22)–(1.28) is now arranged in order to fit with the specific choices for the model formulation.

Equation of motion

$$\frac{d\mathbf{V}}{dt} = -RT\frac{\nabla p}{p} - \frac{\partial p}{\partial \pi}\nabla\phi + \mathcal{V}$$

Equation of $\hat{d}$

The equation for $\hat{d}$ is:

$$\begin{aligned}\frac{d(\hat{d})}{dt} = & - \frac{g^2}{RT^*}\frac{\partial}{\partial \log \pi^*}\left(\frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \pi}\right) + g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w + g\frac{\rho^*}{m^*}\frac{\partial \dot{\eta}}{\partial \eta}.\frac{\partial w}{\partial \eta} \\ & - g\frac{1}{m^*}\,\dot{\eta}\,\frac{\partial w}{\partial \eta}\frac{d\rho^*}{d\eta} + g\frac{\rho^*}{m^{*2}}\,\dot{\eta}\,\frac{\partial w}{\partial \eta}\frac{dm^*}{d\eta} - g\frac{\rho^*}{m^*}\frac{\partial \mathcal{W}}{\partial \eta}\end{aligned}$$

which writes, using $(m\dot{\eta})$:

$$
\begin{aligned}
\frac{d(\hat{d})}{dt} = & - \frac{g^2}{RT^*}\frac{\partial}{\partial \log \pi^*}\left(\frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \pi}\right) + g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w + g\rho^* \frac{\partial (m\dot{\eta})}{\partial \pi^*}\frac{\partial w}{\partial \pi} \\
& - g(m\dot{\eta})\frac{d\rho^*}{d\pi}\frac{\partial w}{\partial \pi^*} + g\rho^*(m\dot{\eta})\frac{\partial}{\partial \pi^*}\left(\frac{\partial w}{\partial \pi}\right) - g\rho^*(m\dot{\eta})\frac{\partial}{\partial \pi}\left(\frac{\partial w}{\partial \pi^*}\right) - g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathcal{W}}{\partial \eta}
\end{aligned}
$$

and in the final form, since $\hat{d} = -g(\pi^*/RT^*)(\partial w/\partial \pi^*)$:

$$
\begin{aligned}
\frac{d(\hat{d})}{dt} = & - \frac{g^2}{RT^*}\frac{\partial}{\partial \log \pi^*}\left(\frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \pi}\right) + g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w - \hat{d}\frac{\partial (m\dot{\eta})}{\partial \pi} \\
& + (m\dot{\eta})\frac{\hat{d}}{\pi^*}\frac{\partial \pi^*}{\partial \pi} - \pi^*(m\dot{\eta})\frac{\partial}{\partial \pi^*}\left(\frac{\partial \pi^*}{\partial \pi}\frac{\hat{d}}{\pi^*}\right) + \pi^*(m\dot{\eta})\frac{\partial}{\partial \pi}\left(\frac{\hat{d}}{\pi^*}\right) - g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathcal{W}}{\partial \eta}
\end{aligned}
$$

Equation of $\hat{\mathcal{P}}$

The equation for $\hat{\mathcal{P}}$ is used in the form:

$$
\frac{d\hat{\mathcal{P}}}{dt} = -\frac{C_p}{C_v}\hat{\mathcal{P}}D_3 - \frac{\pi}{\pi^*}\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right) - \frac{\hat{\mathcal{P}}}{\pi^*}(m\dot{\eta})\frac{\partial \pi^*}{\partial \pi} + \frac{1}{\pi^*}\frac{Qp}{C_v T}
$$

Thermodynamic equation

$$
\frac{dT}{dt} = -\frac{RT}{C_v}D_3 + \frac{Q}{C_v}
$$

Surface pressure tendency equation

$$
\frac{\partial \pi_s}{\partial t} = -\nabla \int_0^1 m\mathbf{V}\, d\eta
$$

Summary of dynamical model prognostic equations

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} &= -RT\frac{\nabla p}{p} - \frac{\partial p}{\partial \pi}\nabla\phi + \mathcal{V} && (1.45)\\
\frac{d(\hat{d})}{dt} &= -\frac{g^2}{RT^*}\frac{\partial}{\partial \log \pi^*}\left(\frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \pi}\right) + g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathbf{V}}{\partial \eta}.\nabla w \\
&\quad -\hat{d}\frac{\partial(m\dot{\eta})}{\partial \pi} + (m\dot{\eta})\frac{\hat{d}}{\pi^*}\frac{\partial \pi^*}{\partial \pi} \\
&\quad -\pi^*(m\dot{\eta})\frac{\partial}{\partial \pi^*}\left(\frac{\partial \pi^*}{\partial \pi}\frac{\hat{d}}{\pi^*}\right) + \pi^*(m\dot{\eta})\frac{\partial}{\partial \pi}\left(\frac{\hat{d}}{\pi^*}\right) \\
&\quad -g\frac{\pi^*}{m^* RT^*}\frac{\partial \mathcal{W}}{\partial \eta} && (1.46)\\
\frac{d\hat{\mathcal{P}}}{dt} &= -\frac{C_p}{C_v}\hat{\mathcal{P}}D_3 - \frac{\pi}{\pi^*}\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right) - \frac{\hat{\mathcal{P}}}{\pi^*}(m\dot{\eta})\frac{\partial \pi^*}{\partial \pi} + \frac{1}{\pi^*}\frac{Qp}{C_v T} && (1.47)\\
\frac{dT}{dt} &= -\frac{RT}{C_v}D_3 + \frac{Q}{C_v} && (1.48)\\
\frac{\partial \pi_s}{\partial t} &= -\nabla\int_0^1 m\mathbf{V}\,d\eta && (1.49)
\end{aligned}
$$

1.5.3 Dynamical model diagnostic relations

Some diagnostic relations are used in order to compute various terms involving the previous prognostic equations:

$$
\begin{aligned}
\mathcal{P} &= \frac{(p-\pi)}{\pi} && (1.50)\\
d &= -g\frac{\pi(1+\mathcal{P})}{mRT}\frac{\partial w}{\partial \eta} && (1.51)\\
\phi &= \phi_s + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})}d\eta && (1.52)\\
D_3 &= \nabla\mathbf{V} + \frac{1}{m}\frac{p}{RT}\nabla\phi.\left(\frac{\partial \mathbf{V}}{\partial \eta}\right) - \frac{g}{m}\frac{p}{RT}\frac{\partial w}{\partial \eta} && (1.53)\\
m\dot{\eta} &= B\int_0^1 \nabla.m\mathbf{V}\,d\eta - \int_0^\eta \nabla.m\mathbf{V}\,d\eta && (1.54)\\
\frac{\dot{\pi}}{\pi} &= \frac{1}{\pi}\left(\mathbf{V}.\nabla\pi - \int_0^\eta \nabla.m\mathbf{V}\,d\eta\right) && (1.55)
\end{aligned}
$$

Chapitre 2

Vertical Discretisation

(01/06/2001)

The choice of the vertical discretisation has some implications on the conservation properties and stability of the model equations. Following Simmons and Burridge (1981), and elementary mathematical conservation properties, we will show in this section and in the next one, that provided a minimal set of arbitrary choices, the various mathematical and physical constraints lead to a unique expression of the discrete vertical scheme.

2.1 Definitions

The vertical domain is discretised in L layers indexed from 1 at the top to L at the bottom. The generic index for these layers is noted l. Each layer is separated from his neighbour by interfaces, the generic index of which is $\tilde{l}$. The interfaces are indexed from 0 to L which respectively represent the upper and lower boundaries of the domain as shown on the figure (??). The grid is vertically staggered such as model variables are defined in the layers, and the vertical fluxes at the interfaces. The vertical velocity is also defined at interfaces, so that its pseudo vertical derivative $\hat{d}$ which is the real prognostic variable of the linear model, is defined in the layers. A variable X defined in the layer l is noted X_l, and at the interface $X_{\tilde{l}}$.

The fact that the vertical coordinate η is implicit results in a direct definition of the hydrostatic pressure π at the interfaces:

$$\pi_{\tilde{l}} = A_{\tilde{l}} + B_{\tilde{l}}.\pi_{\tilde{L}}$$

which is the discrete form of the coordinate definition. The hydrostatic pressure depth of a layer l is:

$$\delta\pi_l = \pi_{\tilde{l}} - \pi_{\tilde{l}-1}$$

2.2 Discretisation of the Linear Model

In the linear model used later for the semi-implicit scheme, the interfaces and thicknesses are taken to their basic state value:

$$\begin{aligned} \pi^*_{\tilde{l}} &= A_{\tilde{l}} + B_{\tilde{l}}.\pi^*_{\tilde{L}} \\ \delta\pi^*_l &= \pi^*_{\tilde{l}} - \pi^*_{\tilde{l}-1} \end{aligned}$$

The discretisation of the linear model is obtained by taking the discrete counterpart of the original linear system (1.37)-(1.41). For this we define the discrete operators corresponding to the continuous integral and derivative operators introduced in section 1.4.1:

$$\begin{aligned}
(\mathbf{G}^*.X)_l &= \left[\sum_{k=l+1}^{L} \delta_k^* X_k + \alpha_l^* X_l\right] \\
(\mathbf{S}^*.X)_l &= \left[\frac{1}{\pi_l^*}\sum_{k=1}^{l-1} \delta\pi_k^* X_k + \beta_l^* X_l\right] \\
(\mathbf{N}^*.X)_l &= \frac{1}{\pi_{\tilde{L}}^*}\sum_{k=1}^{L} \delta\pi_k^* X_k
\end{aligned}$$

where π_k^* is a representative basic state hydrostatic pressure in the layer k, and δ_k^* a discrete form for $d(\log \pi^*)$ in the layer k, to be determined later; the quantities α_l^* and β_l^* are two dimensionless shifts for completing the discrete integrals from the closest interface level to the layer l level. The four latter quantities are, for the moment, arbitrary, and will be chosen conveniently to insure conservation properties. The discrete second vertical derivative operator representing the discrete counterpart of $\mathcal{L}^*$ is noted $\mathbf{L}^*$, but no special shape is *a priori* retained for it.
The starting point is the linear system (1.37) – (1.41), which formally conserve the same shape in discrete form, provided that each continuous vertical operator (i.e: $\mathcal{L}^*$, $\mathcal{G}^*$,...) is replaced by its discrete equivalent (i.e: $\mathbf{L}^*$, $\mathbf{G}^*$,...). Then it can be seen that the derivation is formally identical in discrete and continuous case, and a similar operator $\mathbf{A_1^*}$ appears in the discrete equivalent of the divergence equation (1.42). This operator writes:

$$\mathbf{A_1^*} = -\mathbf{G}^*\mathbf{S}^* + \mathbf{G}^* + \mathbf{S}^* - \mathbf{N}^*$$

we thus define a first constaint (C1) for the discrete operators by imposing that, as in the continuous case:

$$\underline{\text{(C1) :}} \qquad \mathbf{A_1^*} = 0 \tag{2.1}$$

The elimination then can be pursued identically to the continuous one, until the derivation of the final structure equation. It should be noted that with the elimination order chosen here, no commutativity property has been used to derive the structure equation. The derivation thus leads to another constraint (C2) concerning the operator applying on $\Delta\hat{d}$:

$$\underline{\text{(C2) :}} \qquad g^2\mathbf{L}^*.\mathbf{A_2^*} = g^2\mathbf{L}^*\left(\mathbf{S}^*\mathbf{G}^* - \frac{C_p}{C_v}\mathbf{S}^* - \frac{C_p}{C_v}\mathbf{G}^*\right) = N^2c^2 \tag{2.2}$$

The problem is to define the four unknowns α_l^*, β_l^*, π_l^*, δ_l^* such as these two constraints are fullfilled.

2.3 Determination of Discrete Vertical Operators in the Linear Model

2.3.1 First constraint

In order to examine the first constaint, we write the matrix of the $\mathbf{A_1^*}$ operator. We respectively note sup$_{(i,j)}$, inf$_{(i,j)}$ and diag$_{(i,i)}$, the generic superior, inferior, and diagonal term of the matrix for the row i and the column j. The operator $\mathbf{A_1^*}$ then writes:

$$
\begin{aligned}
\mathsf{sup}_{(i,j)} &= \delta_j^*(1-\beta_j^*) - \delta\pi_j^* \left(\frac{1}{\pi_{\tilde{L}}^*} + \sum_{k=j+1}^{L} \frac{\delta_k^*}{\pi_k^*} \right) \\
\mathsf{diag}_{(i,i)} &= \alpha_i^* + \beta_i^* - \alpha_i^*\beta_i^* - \delta\pi_i^* \left(\frac{1}{\pi_{\tilde{L}}^*} + \sum_{k=i+1}^{L} \frac{\delta_k^*}{\pi_k^*} \right) \\
\mathsf{inf}_{(i,j)} &= \frac{\delta\pi_j^*}{\pi_i^*}(1-\alpha_i^*) - \delta\pi_j^* \left(\frac{1}{\pi_{\tilde{L}}^*} + \sum_{k=i+1}^{L} \frac{\delta_k^*}{\pi_k^*} \right)
\end{aligned}
$$

The solution of this system leads to a first set of constaint between the four unknowns:

$$
\begin{aligned}
\alpha_l^* &= 1 - \frac{\pi_l^*}{\pi_{\tilde{l}}^*} \\
\beta_l^* &= 1 - \frac{\pi_{\tilde{l}-1}^*}{\pi_l^*} \\
\frac{\delta_l^*}{\pi_l^*} &= \frac{1}{\pi_{\tilde{l}-1}^*} - \frac{1}{\pi_{\tilde{l}}^*}
\end{aligned}
$$

2.3.2 Second constraint

The second constraint consists in identifying $g^2\mathbf{L}^*.\mathbf{A_2^*}$ to a N^2c^2 homothecy. The natural discrete operator for a double vertical derivative should be tridiagonal in order to reflect the "locality" of the continuous version. The weight of the three levels concerned by a given layer l in the $\mathbf{L}^*$ operator will be noted A_l^*, B_l^* and C_l^*; thus we have:

$$
(\mathbf{L}^*.X)_l = \mathsf{A}_l^*.X_{l-1} + \mathsf{B}_l^*.X_l + \mathsf{C}_l^*.X_{l+1}
$$

We proceed as for the first constraint, writing the matrix form of $\mathbf{A_2^*}$.

$$
\begin{aligned}
\mathsf{sup}_{(i,j)} &= -\frac{R}{C_v}\delta_j^* \\
\mathsf{diag}_{(i,i)} &= -\frac{R}{C_v}(\alpha_i^* + \beta_i^*) \\
\mathsf{diag}_{(1,1)} &= -\frac{R}{C_v}(\alpha_1^* + \beta_1^*) + 1 \\
\mathsf{inf}_{(i,j)} &= -\frac{R}{C_v}\frac{\delta\pi_j^*}{\pi_i^*}
\end{aligned}
$$

It can be shown that provided the first constraint is fullfilled, the matrix $\mathbf{L}^*.\mathbf{A_2^*}$ $cannot$ be identified to an homothecy, since it would require $\alpha_l^* + \beta_l^* = \delta_l^*$, i.e. $\pi_l^* = \pi_{\tilde{l}}^*$, which means no more corrective shift for evaluating the integral operators in the layers. This solution should not be retained since it acts as a non-centered vertical scheme.

One can further show that requiring this second matrix to be purely diagonal is even incompatible with any convenient non trivial choice of the four unknowns, independently of whatever or not the first constraint is fullfilled.

On the other hand it is sufficient for our purpose that the operator $\mathbf{T}^* = (g^2/N^2c^2)\mathbf{L}^*.\mathbf{A_2^*}$ should be tridiagonal (instead of purely diagonal), provided this operator behaves physically as a local vertical average operator applying to $\hat{d}$ values.

The final form of the second constraint then becomes:

$\underline{(\mathrm{C2}'):}$ $$g^2\mathbf{L}^*.\mathbf{A_2^*} = g^2\mathbf{L}^*\left(\mathbf{S}^*\mathbf{G}^* - \frac{C_p}{C_v}\mathbf{S}^* - \frac{C_p}{C_v}\mathbf{G}^*\right) = N^2c^2\mathbf{T}^* \quad (2.3)$$

The resolution of the constraint is written by imposing the weight of this local averaging to be equal to 1. The physical way to impose this is to consider the vertical shift in vertical velocity induced by an individual $\hat{d}_l$ value and by its "averaged" counterpart $(\mathbf{T}.\hat{d})_l$. This shift must be equal in the two cases, in order to conserve the advective properties of the vertical velocity field. Writing equality between these two shifts leads to:

$$\delta w_l' = -H\frac{m^*}{\pi^*}\hat{d}_l = -H\frac{m^*}{\pi^*}\left(\mathbf{T}^*\hat{d}\right)_l$$

which expresses the physical normalisation constraint applying on the averaging operator $\mathbf{T}^*$.
After some algebraic manipulation, one shows that all the above constraints are fullfilled for the following solution:

$$\begin{aligned}
\alpha_l^* &= \beta_l^* = 1 - \sqrt{\frac{\pi_{\tilde{l}-1}^*}{\pi_{\tilde{l}}^*}} \\
\pi_l^* &= \sqrt{\pi_{\tilde{l}-1}^*.\pi_{\tilde{l}}^*} = \frac{\delta\pi_l^*}{\delta_l^*} \\
\delta_l^* &= \frac{\delta\pi_l^*}{\pi_l^*} \\
\mathsf{A}_l^* &= \frac{\pi_{l-1}^*}{\delta_l^*\left(\pi_l^* - \pi_{l-1}^*\right)} \\
\mathsf{B}_l^* &= -\frac{1}{\delta_l^*}\left(\frac{\pi_l^*}{\pi_l^* - \pi_{l-1}^*} + \frac{\pi_l^*}{\pi_{l+1}^* - \pi_l^*}\right) \\
\mathsf{C}_l^* &= \frac{\pi_{l+1}^*}{\delta_l^*\left(\pi_{l+1}^* - \pi_l^*\right)}
\end{aligned}$$

The determination of the values at the top gives:

$$\begin{aligned}
\alpha_1^* &= \beta_1^* = 1 \\
\delta_1^* &= \alpha_1^* + \beta_1^* + \frac{C_v}{R} = 2 + \frac{C_v}{R} = 1 + \frac{C_p}{R} \\
\pi_1^* &= \frac{\delta\pi_1^*}{\delta_1^*} \\
\mathsf{A}_1^* &= 0 \\
\mathsf{C}_1^* &= \frac{\pi_2^*}{\delta_1^*\left(\pi_2^* - \pi_1^*\right)} \\
\mathsf{B}_1^* &= -\mathsf{C}_1^*
\end{aligned}$$

The bottom of the domain is not really a particular case provided that $\mathsf{C}_L^* = 0$ is taken. Then:

$$\begin{aligned}
\mathsf{A}_L^* &= \frac{\pi_{L-1}^*}{\delta_L^*\left(\pi_L^* - \pi_{L-1}^*\right)} \\
\mathsf{B}_L^* &= -\frac{\pi_L^*}{\delta_L^*\left(\pi_L^* - \pi_{L-1}^*\right)}
\end{aligned}$$

For $l \in [1, L-1]$, the general expression for $\mathbf{L}^*$ can be rewritten, after a simplification:

$$\mathbf{L}^*.X_l = \mathsf{A}_l^*(X_{l-1} - X_l) + \mathsf{C}_l^*(X_{l+1} - X_l) \quad (2.4)$$

The tridiagonal $\mathbf{T}^*$ matrix elements write, for the row i:

$$
\begin{array}{lrcl}
\text{(diag)} & \mathbf{T}^*_{(i,i)} & = & 1 - \frac{1}{\delta_i^*}\left(\frac{\pi_i^*}{\pi_{i+1}^* - \pi_i^*} + \frac{\pi_i^*}{\pi_i^* - \pi_{i-1}^*}\right)(\delta_i^* - 2\alpha_i^*) \\
\text{(diag)} & \mathbf{T}^*_{(1,1)} & = & 1 \\
\text{(inf)} & \mathbf{T}^*_{(i,i-1)} & = & \frac{1}{\delta_i^*}\frac{\pi_{i-1}^*}{\pi_i^* - \pi_{i-1}^*}\left(\delta_{i-1}^* - 2\alpha_{i-1}^*\right) \\
\text{(sup)} & \mathbf{T}^*_{(i,i+1)} & = & \frac{1}{\delta_i^*}\frac{\pi_{i+1}^*}{\pi_{i+1}^* - \pi_i^*}\left(\delta_{i+1}^* - 2\alpha_{i+1}^*\right)
\end{array}
$$

which can finally be written:

$$
\mathbf{T}^* = (\mathbf{I} + \mathbf{L}^*.\mathbf{Q}^*) \tag{2.5}
$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{Q}^*$ is the diagonal matrix the row i diagonal element of which is given by $\mathbf{Q}^*_{(i,i)} = (\delta_i^* - 2\alpha_i^*)$ for $i \in [2, L]$, and $\mathbf{Q}^*_{(1,1)} = 0$.

2.3.3 Structure equation

The obtained structure equation is then for the discrete case:

$$
\left[-\frac{1}{c^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(\Delta + \frac{\mathbf{L}^*}{H^2}\right) + N^2\mathbf{T}^*\Delta\right]\hat{d} = 0 \tag{2.6}
$$

We now clearly see the "local" nature of the structure equation which is characteristic of the relaxation of hydrostatic "global" effects. The obtained equation results in a tridiagonal inversion with a constant coefficients matrix, given the time step and the horizontal total wavenumber. This means that, like in the hydrostatic case, the inverse of the semi-implicit operator can be pre-computed for each total wavenumber, and stored at the beginning of the model integration.

The major difference between the continuous and discrete structure equations is the appearance of the $\mathbf{T}^*$ operator. The properties of this operator are now examined. The diagonal term of the $\mathbf{Q}^*$ matrix can be rewritten as:

$$
(\delta_l^* - 2\alpha_l^*) = \frac{\left(\sqrt{\pi_{\tilde{l}}^*} - \sqrt{\pi_{\tilde{l}-1}^*}\right)^2}{\sqrt{\pi_{\tilde{l}-1}^*\pi_{\tilde{l}}^*}} = \frac{\delta\pi_l^2}{\sqrt{\pi_{\tilde{l}-1}^*\pi_{\tilde{l}}^*}\left(\sqrt{\pi_{\tilde{l}}^*} + \sqrt{\pi_{\tilde{l}-1}^*}\right)^2} = \frac{\delta_l^2}{4}\frac{2}{1 + \sqrt{1 + \frac{\delta_l^2}{4}}}
$$

We see that all elements of the diagonal matrix $\mathbf{Q}^*$ are positive (or zero for $\mathbf{Q}^*_{(1,1)}$) and smaller than $\delta_l^2/4$. When the resolution becomes thin enough, the latter expression tends toward its continuous counterpart :

$$
(\delta_l^* - 2\alpha_l^*) \simeq \delta_l^2/4
$$

Similarly, at the limits of high and regular vertical resolutions, the internal expression of the $\mathbf{L}^*$ matrix becomes:

$$
A_l^* = C_l^* = -B_l^*/2 = 1/\delta_l^2
$$

which yields for the matrix $\mathbf{L}^*\mathbf{Q}^*$:

$$
\mathbf{L}^*\mathbf{Q}^* \quad \longrightarrow \quad \frac{1}{4}\mathbf{D}_2^*
$$

where $\mathbf{D}_2^*$ is the canonical second order finite-difference (tridiagonal and symmetric) operator whose general row is (0 ... 0, 1, -2, 1, 0 ... 0).

It can be seen that even when the number of levels tend towards infinity, the additional $\mathbf{T}^*$ matrix does not tends towards zero, and the discrete systems thus does not formally converge towards the continuous one. This problem will be addressed in the next chapter on time discretisation.

2.4 Discretisation of the Non-Linear Model

The discretisation of the complete non-linear model follows the rules edicted for the linear model. In particular, each discretisation parameter in the linear model has its own counterpart in the complete model.

2.4.1 Vertical levels

The discrete vertical level characteristics are taken under the same form as in the linear model:

$$
\begin{aligned}
\pi_l &= \sqrt{\pi_{\tilde{l}-1}.\pi_{\tilde{l}}} = \frac{\delta\pi_l}{\delta_l} \\
\delta_l &= \frac{\delta\pi_l}{\pi_l}
\end{aligned}
$$

with the special boundary condition at top:

$$
\begin{aligned}
\delta_1 &= \alpha_1 + \beta_1 + \frac{C_v}{R} = 2 + \frac{C_v}{R} = 1 + \frac{C_p}{R} \\
\pi_1 &= \frac{\delta\pi_1}{\delta_1}
\end{aligned}
$$

Similarly, the definition of the discretization parameters α_l and β_l is taken from the linear form:

$$
\alpha_l = \beta_l = 1 - \sqrt{\frac{\pi_{\tilde{l}-1}}{\pi_{\tilde{l}}}}
$$

with the special boundary condition at top:

$$
\alpha_1 = \beta_1 = 1
$$

2.4.2 Vertical operators

For the vertical integral operators, the same principle is adopted than in the linear model to perform summation from a boundary to the level of the layer l, by addition of a corrective term α_l:

$$
\left(\int_\eta^1 \frac{m}{\pi} X d\eta\right)_l = \sum_{k=l+1}^{L} X_k\delta_k + \alpha_l X_l \qquad (2.7)
$$

$$
\left(\frac{1}{\pi}\right)\left(\int_0^\eta mX d\eta\right)_l = \frac{1}{\pi}\sum_{k=0}^{l-1} X_k\delta\pi_k + \alpha_l X_l \qquad (2.8)
$$

$$\left(\int_0^1 mX\,d\eta\right)_l = \sum_{k=1}^{L} X_k \delta\pi_k \tag{2.9}$$

Strictly speaking, the second vertical derivative analogous to $\mathbf{L}^*$ in the linear model (which could be defined as $\mathbf{L}$, by removing all stars in the definition of $\mathbf{L}^*$) is *not used* in the non-linear model, as it can be seen in Eq. (1.45) – (1.49).
On the other hand, Eq. (1.46) shows that the following $\mathbf{L}'$ operator appears, applied to $\hat{\mathcal{P}}$:

$$[\mathbf{L}'X]_l = \left[\pi^* \frac{\partial}{\partial \pi^*}\left(\frac{\partial \pi^* X}{\partial \pi}\right)\right]_l = \mathsf{A}'_l X_{l-1} + \mathsf{B}'_l X_l + \mathsf{C}'_l X_{l+1}$$

Here also, the definition for the coefficients of $\mathbf{L}'$ is chosen to reflect the ones of the linear model:

$$\begin{aligned}
\mathsf{A}'_l &= \frac{\pi^*_{l-1}}{\delta^*_l\left(\pi_l - \pi_{l-1}\right)} \\
\mathsf{B}'_l &= -\frac{1}{\delta^*_l}\left(\frac{\pi^*_l}{\pi_l - \pi_{l-1}} + \frac{\pi^*_l}{\pi_{l+1} - \pi_l}\right) \\
\mathsf{C}'_l &= \frac{\pi^*_{l+1}}{\delta^*_l\left(\pi_{l+1} - \pi_l\right)}
\end{aligned}$$

with the following boundary condition at top:

$$\begin{aligned}
\mathsf{A}'_1 &= 0 \\
\mathsf{C}'_1 &= \frac{\pi^*_2}{\delta^*_1\left(\pi_2 - \pi_1\right)} \\
\mathsf{B}'_1 &= -\mathsf{C}'_1
\end{aligned}$$

Unlike in Eq.(2.4), the general expression for $\mathbf{L}'$ cannot be simplified to involve only A' and C'.

2.4.3 Bottom Boundary Condition for the Non-Linear Laplacian Operator $\mathbf{L}'$

The definition of the bottom boundary condition for $\mathbf{L}'$ is a little bit more complicated than for the top, and is based on the w equation.
For $l = L$ we have:

$$\left[\mathbf{L}'(\hat{\mathcal{P}})\right]_L = \frac{1}{\delta^*_L}\left[-\left(\frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \pi}\right)_{\tilde{L}-1} + \left(\frac{\partial \pi^* \hat{\mathcal{P}}}{\partial \pi}\right)_s\right]$$

We now must find an expression for the last term. First, it should be noted from Eq.(1.13) that:

$$\dot{w} = \mathcal{W} + g\frac{\partial\left(p - \pi\right)}{\partial \pi}$$

thus:

$$g\frac{\partial}{\partial \pi}\left(\pi^* \hat{\mathcal{P}}\right)_s = \dot{w}_s - \mathcal{W}_s$$

A parcel located at ground will indefinitely stay at ground since $\dot{\eta}(1) = 0$. Therefore, for such a parcel the quantity $\dot{w}_s$ is meaningful and is only governed by the flow parallel to the surface $\mathbf{V}_s$:

$$g w_s = \mathbf{V}_s.\nabla\phi_s \qquad \Rightarrow \qquad g\dot{w}_s = \frac{\partial \mathbf{V}_s}{\partial t}.\nabla\phi_s + \mathbf{V}_s.\nabla\left(\mathbf{V}_s.\nabla\phi_s\right)$$

or:

$$g\dot{w}_s = \dot{\mathbf{V}}_s.\nabla\phi_s + \left(u_s^2\frac{\partial^2\phi_s}{\partial x^2} + 2u_s v_s\frac{\partial^2\phi_s}{\partial x\partial y} + v_s^2\frac{\partial^2\phi_s}{\partial y^2}\right)$$

We note J_s the last bracketed Jacobian surface term. The final expression for $\left[\mathbf{L}'(\hat{\mathcal{P}})\right]_L$ is then:

$$\left[\mathbf{L}'(\hat{\mathcal{P}})\right]_L = \frac{-1}{\delta_L^*}\left[\left(\frac{\pi_L^*\hat{\mathcal{P}}_L - \pi_{L-1}^*\hat{\mathcal{P}}_{L-1}}{\pi_L - \pi_{L-1}}\right) - \frac{1}{g^2}\left(\dot{\mathbf{V}}_s.\nabla\phi_s + J_s\right) + \frac{1}{g}\mathcal{W}_s\right] \tag{2.10}$$

In practice for the model, it is assumed that the boundary condition problem is internal to the dynamical package, and thus $\mathcal{W}$ is not considered. Consistently, $\dot{\mathbf{V}}_s$ is supposed to only contain Coriolis, pressure-gradient and curvature terms (but no physical contribution).

2.5 Expression of w related terms

We give here the discretised expressions for gw and $g\nabla w$. The term $g\nabla w$ is needed for $\hat{d}$ evolution in Eq. (1.46) while gw is not used for evolution but its expression is given for information.
From Eq. (1.33), the continuous expression for gw and $g\nabla w$ are:

$$\begin{aligned} gw &= gw_s + \int_\eta^1 \frac{m^* RT^*}{\pi^*}\hat{d}.d\eta \\ g\nabla w &= g\nabla w_s + \int_\eta^1 \frac{m^* RT^*}{\pi^*}\nabla\hat{d}.d\eta \end{aligned}$$

with $gw_s = \mathbf{V}_s.\nabla\phi_s$ and $g\nabla w_s = \nabla(\mathbf{V}_s.\nabla\phi_s)$ the surface vertical velocity value and horizontal gradient.
In discrete form, $\mathbf{V}_s$ is taken to $\mathbf{V}_L$, thus $gw_s = \mathbf{V}_L.\nabla\phi_s$ and $g\nabla w_s = \nabla(\mathbf{V}_L.\nabla\phi_s)$
Finally:

$$\begin{aligned} gw_{\tilde{l}} &= \mathbf{V}_L.\nabla\phi_s + \sum_{k=l+1}^{L} RT^*\delta^*\hat{d}_k \\ g\nabla w_{\tilde{l}} &= \nabla\left(\mathbf{V}_L.\nabla\phi_s\right) + \sum_{k=l+1}^{L} RT^*\delta^*\nabla\hat{d}_k \end{aligned} \tag{2.11}$$

2.6 Expression of non-advective $\frac{\partial\mathbf{V}}{\partial\eta}$ terms

Examination of the original system shows that two non-advective terms involving $(\partial\mathbf{V}/\partial\eta)$, namely in Eq. (1.46) and (1.53) This kind of term is problematic since $\delta\mathbf{V}$ is naturally available only at levels $\tilde{l}$.
The evolution term involving $[g(\partial\mathbf{V}/\partial\eta).\nabla w]$ in Eq.(1.46) can be rewritten as $[(\delta\mathbf{V}.g\nabla w)_{\tilde{l}}\ /\ (RT^*\delta_{\tilde{l}}^*)]$. It can then be obtained by linearly interpolating the horizontal wind between vertical levels in the layers:

$$\mathbf{V}_{\tilde{l}} = \epsilon_l\mathbf{V}_l + (1-\epsilon_l)\mathbf{V}_{l+1} \tag{2.12}$$

with ϵ_l being the weights for linear interpolator:

$$\epsilon_l = \frac{\delta_{l+1} - \alpha_{l+1}}{\delta_{l+1} - \alpha_{l+1} + \alpha_l} \tag{2.13}$$

The final form of the evolution term is:

$$\frac{(\delta \mathbf{V}.g\nabla w)_l}{RT^*\delta_l^*} = \frac{1}{RT^*\delta_l^*}\left[\left(\mathbf{V}_{\tilde{l}} - \mathbf{V}_l\right)g\nabla w_{\tilde{l}} + \left(\mathbf{V}_l - \mathbf{V}_{\widetilde{l-1}}\right)g\nabla w_{\widetilde{l-1}}\right]$$

Exactely the same process is applied for the term involving $[(\partial \mathbf{V}/\partial\eta).\nabla\phi]$ in Eq.(1.53).

The top and bottom boundary condition for these terms are formulated by assuming a free-slip type condition. It appears that the stability of the model is *very* sensitive to the formulation of the bottom BC of the previous term in the D_3 diagnostic equation. For the moment, a rather "heuristic" formulation which ensure stability has been found, but some extra-work is needed to find a more satisfactorily formulation.
The formulation which ensures stability consists in using the linearly interpolated values of half-level winds given by Eq. (2.12) to specifiy artificial values $\mathbf{V'}_{\tilde{l}}$ near the boundaries:

$$\begin{aligned}
\mathbf{V'}_{\tilde{3}} &= \left(2\mathbf{V}_{\tilde{3}} + \mathbf{V}_{\tilde{2}}\right)/3 \\
\mathbf{V'}_{\tilde{2}} &= \left(2\mathbf{V}_{\tilde{1}} + \mathbf{V}_{\tilde{2}}\right)/3 \\
\mathbf{V'}_{\tilde{1}} &= \mathbf{V}_1 \\
\mathbf{V'}_{\tilde{0}} &= \mathbf{V}_1
\end{aligned}$$

$$\begin{aligned}
\mathbf{V'}_{\widetilde{L-3}} &= \left(2\mathbf{V}_{\widetilde{L-3}} + \mathbf{V}_{\widetilde{L-2}}\right)/3 \\
\mathbf{V'}_{\widetilde{L-2}} &= \left(2\mathbf{V}_{\widetilde{L-1}} + \mathbf{V}_{\widetilde{L-2}}\right)/3 \\
\mathbf{V'}_{\widetilde{L-1}} &= \mathbf{V}_L \\
\mathbf{V'}_{\tilde{L}} &= \mathbf{V}_L
\end{aligned}$$

2.7 Conservation of Energy by the Vertical Scheme

Laprise showed that the total energy invariant on the sphere is defined, for the considered system by:

$$\int_0^{2\pi}\int_{-\pi/2}^{-\pi/2}\int_0^1 (K + C_vT + \phi)m.d\eta.a\cos\varphi.d\varphi.d\lambda$$

where λ and φ are the longitude and latitude, and K is the 3-dimensional kinetic energy: $K = (1/2)(\mathbf{V}^2 + w^2)$. The conservation of this total energy by the vertical advective scheme leads to an imposed form for the vertical advection terms. Following Simmons and Burridge, 1981, the preservation by the finite-difference scheme of energy conservation is achieved if for any variable F, the vertical integral of the advection term verifies the *per partes* following property, given the fact that $\dot{\eta} = 0$ at the boundaries of the domain:

$$\begin{aligned}
\int_0^1 m\dot{\eta}\frac{\partial F}{\partial\eta}d\eta &= -\int_0^1 F\frac{\partial}{\partial\eta}(m\dot{\eta})d\eta \\
\int_0^1 m\dot{\eta}F\frac{\partial F}{\partial\eta}d\eta &= -\int_0^1 \frac{F^2}{2}\frac{\partial}{\partial\eta}(m\dot{\eta})d\eta
\end{aligned}$$

This property is true if the vertical advection term is chosen as:

$$\left(\dot{\eta}\frac{\partial F}{\partial\eta}\right)_l = \frac{1}{2\delta\pi_l}\left[(m.\dot{\eta})_{\tilde{l}}\left(F_{l+1} - F_l\right) + (m.\dot{\eta})_{\widetilde{l-1}}\left(F_l - F_{l-1}\right)\right]$$

The advective term thus takes the same form as in hydrostatic case with true pressure. This allows to keep the form of the advective term already taken in ARPEGE/aladin system.

2.8 Conservation of Angular Momentum by the Vertical Scheme

2.8.1 Continuous form of the constraint

In an hydrostatic atmosphere, the conservation of angular momentum in the absence of external forcing can be expressed on a latitude circle (Simmons and Burridge, 1981):

$$\int_0^{2\pi}\left[\int_0^1\left(\frac{\partial\phi}{\partial\lambda}+\frac{RT}{\pi}\frac{\partial\pi}{\partial\lambda}\right)m.d\eta+\phi_s\frac{\partial\pi_s}{\partial\lambda}\right]d\lambda=0$$

In the case of fully compressible fluid, this conservation law must be revised. First, the left parenthesis term, which represents the full pressure force, must be expressed now in terms of *true* pressure, as well as the last term, which represent the dynamical drag by the planet surface on the fluid. However, the factor $m.d\eta$ must be kept in hydrostatic pressure formulation as in the hydrostatic case, since it really represents the *mass* of the infinitesimal layer. The conservation law then becomes:

$$\int_0^{2\pi}\left[\int_0^1\left(\frac{1}{m}\frac{\partial p}{\partial\eta}\frac{\partial\phi}{\partial\lambda}+\frac{RT}{p}\frac{\partial p}{\partial\lambda}\right)m.d\eta+\phi_s\frac{\partial p_s}{\partial\lambda}\right]d\lambda=0$$

which can be rewritten (since $p_s=\int_0^1(\partial p/\partial\eta)d\eta$):

$$\int_0^{2\pi}\int_0^1\left[\frac{\partial\phi}{\partial\lambda}\frac{\partial p}{\partial\eta}+m\frac{RT}{p}\frac{\partial p}{\partial\lambda}+\phi_s\frac{\partial}{\partial\lambda}\left(\frac{\partial p}{\partial\eta}\right)\right]d\eta\,d\lambda=0$$

Latitude circles are closed trajectories, so:

$$\int_0^{2\pi}\int_0^1\left[-(\phi-\phi_s)\frac{\partial}{\partial\lambda}\left(\frac{\partial p}{\partial\eta}\right)+m\left(\frac{RT}{p}\frac{\partial p}{\partial\lambda}\right)\right]d\eta\,d\lambda=0$$

2.8.2 Constraint for the pressure gradient

One can see that the constraint is formally similar than the constraint in the hydrostatic case (Simmons and Burridge). We introduce here a definition for the discretised version of $(\partial p/\partial\eta)$ at level l:

$$\left(\frac{\partial p}{\partial\eta}\right)_l=(\delta p)_l$$

The conservation of angular momentum in the finite difference scheme is therefore insured if:

$$\begin{aligned}\sum_{l=1}^{L}\left(RT\frac{\nabla p}{p}\right)_l\delta\pi_l &= \sum_{l=1}^{L}(\phi_l-\phi_s)\,\nabla(\delta p)_l\\ &= \sum_{l=1}^{L}\left(\alpha_l\frac{R_lT_l}{1+\mathcal{P}_l}+\sum_{k=l+1}^{L}\frac{R_kT_k\delta k}{1+\mathcal{P}_k}\right)\nabla(\delta p)_l\\ &= \sum_{l=1}^{L}\alpha_l\frac{R_lT_l}{1+\mathcal{P}_l}\nabla(\delta p)_l+\sum_{k=1}^{L}\left(\frac{R_kT_k}{1+\mathcal{P}_k}\delta_k\sum_{l=1}^{k-1}\nabla(\delta p)_l\right)\end{aligned}$$

This leads to the discrete form of the non-hydrostatic pressure gradient term imposed by the conservation of angular momentum:

$$\left(RT\frac{\nabla p}{p}\right)_l=\frac{R_lT_l}{1+\mathcal{P}_l}\frac{1}{\delta\pi_l}\left(\alpha_l\nabla(\delta p)_l+\delta_l\sum_{k=1}^{l-1}\nabla(\delta p)_k\right)$$

It is now obvious that this constraint imposes the shape of $(\delta p)_l$ in order to maintain the *local* character of the expression of the non-hydrostatic pressure gradient: In the fully compressible system described here, the horizontal pressure gradient must not depend on the state of the atmosphere at others levels. This expresses the fact that $(\delta p)_l$ *must* have the form of a difference in order to insure cancellations trough vertical summations in the right hand side term. Returning to the signification of $(\delta p)_l$, we can write:

$$(\delta p)_l = p_{\tilde{l}} - p_{\tilde{l}-1}$$

Any discretisation of $(\partial p/\partial\eta)$ which respects this form will insure the conservation of angular momentum and locality of the non-hydrostatic pressure gradient. The last stage is thus to choose a discrete form for $p_{\tilde{l}}$. Starting from $p = \pi + \pi^*\mathcal{P}$, and according to the fact that the prognostic variable is $\hat{\mathcal{P}}$, the most "natural" form for discretising $p_{\tilde{l}}$ is to express $\hat{\mathcal{P}}$ at interfaces (which writes $\bar{\hat{\mathcal{P}}}$) as an average, defining:

$$\bar{\hat{\mathcal{P}}}_{\tilde{l}} = \left(\frac{\hat{\mathcal{P}}_{l+1} + \hat{\mathcal{P}}_l}{2}\right)$$

This yields:

$$\delta p_l = \delta\pi_l + \pi^*_{\tilde{l}}\bar{\hat{\mathcal{P}}}_{\tilde{l}} - \pi^*_{\tilde{l}-1}\bar{\hat{\mathcal{P}}}_{\tilde{l}-1} = \delta\pi_l + \pi^*_{\tilde{l}}\left(\frac{\hat{\mathcal{P}}_{l+1} + \hat{\mathcal{P}}_l}{2}\right) - \pi^*_{\tilde{l}-1}\left(\frac{\hat{\mathcal{P}}_l + \hat{\mathcal{P}}_{l-1}}{2}\right)$$

At the bottom of the domain, we simply chose:

$$\bar{\hat{\mathcal{P}}}_{\tilde{L}} = \hat{\mathcal{P}}_L$$

At the top of the domain, we chose:

$$\bar{\hat{\mathcal{P}}}_{\tilde{0}} = \hat{\mathcal{P}}_1$$

Finally, the finite difference form for the total pressure gradient term insuring the conservation of angular momentum is:

$$\left(\frac{RT}{p}\nabla p\right)_l = \frac{R_l T_l}{1+\mathcal{P}_l}\frac{1}{\delta\pi_l}\left(\alpha_l\nabla\left(\delta p_l\right) + \delta_l\nabla p_{\tilde{l}-1}\right) \tag{2.14}$$

with the following notations:

$$\begin{aligned} p_{\tilde{l}} &= \pi_{\tilde{l}} + \pi^*_{\tilde{l}}\bar{\hat{\mathcal{P}}}_{\tilde{l}} = \pi_{\tilde{l}} + \pi^*_{\tilde{l}}\left(\frac{\hat{\mathcal{P}}_{l+1} + \hat{\mathcal{P}}_l}{2}\right) \\ \delta p_l &= p_{\tilde{l}} - p_{\tilde{l}-1} \end{aligned}$$

The part involving π can be simplified using the coordinate definition:

$$\frac{R_l T_l}{1+\mathcal{P}_l}\frac{1}{\delta\pi_l}\left(\alpha_l\nabla\delta\pi_l + \delta_l\nabla\pi_{\tilde{l}-1}\right) = \frac{R_l T_l}{1+\mathcal{P}_l}\frac{1}{\delta\pi_l}\left(\frac{\delta_l}{\delta\pi_l}C_l + \delta B_l\right)\nabla\pi_s$$

with:

$$C_l = A_{\tilde{l}}B_{\tilde{l}-1} - A_{\tilde{l}-1}B_{\tilde{l}}$$

This form reverts to those found for $(\nabla\pi/\pi)$ in the hydrostatic case (see Simmons and Burridge).

2.8.3 Expression of the $\nabla\phi$ term

The discretised diagnostic relation for the geopotential is:

$$\phi_l = \phi_s + \sum_{k=l+1}^{L} \left(\frac{R_k T_k}{1+\mathcal{P}_k} \delta_k \right) + \alpha_l \frac{R_l T_l}{1+\mathcal{P}_l}$$

The gradient then writes:

$$\begin{aligned}
(\nabla\phi)_l &= \nabla\phi_s + \sum_{k=l+1}^{L} \left[\frac{R_k T_k}{1+\mathcal{P}_k} \nabla\delta_k + \frac{\nabla(RT)_k}{1+\mathcal{P}_k}\delta_k - \frac{\delta_k}{(1+\mathcal{P}_k)} \left(\frac{RT}{1+\mathcal{P}} \nabla\mathcal{P} \right)_k \right] \\
& \quad + \frac{\alpha_l}{1+\mathcal{P}_l} \nabla(RT)_l + \frac{R_l T_l}{1+\mathcal{P}_l} \nabla\alpha_l - \frac{\alpha_l}{(1+\mathcal{P}_l)} \left(\frac{RT}{1+\mathcal{P}} \nabla\mathcal{P} \right)_l
\end{aligned}$$

The quantity $(\nabla\mathcal{P})_l$ can be related to already computed gradients:

$$\begin{aligned}
(\nabla\mathcal{P})_l &= \frac{\pi_l^*}{\pi_l} \nabla\hat{\mathcal{P}}_l - \mathcal{P}_l \left(\frac{\nabla\pi}{\pi} \right)_l \\
&= \frac{\pi_l^*}{\pi_l} \nabla\hat{\mathcal{P}}_l - \frac{\mathcal{P}_l}{1+\mathcal{P}_l} \frac{1}{\delta\pi_l} \left(\frac{\delta_l}{\delta\pi_l} C_l + \delta B_l \right) \nabla\pi_s
\end{aligned}$$

The horizontal gradients of α and δ, can also be expressed in term of hydrostatic surface preesure horizontal gradient after some algebra:

$$\begin{aligned}
\nabla\alpha_l &= -\frac{1}{2} . \frac{(1-\alpha_l)C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}} \nabla\pi_s \\
\nabla\delta_l &= -\frac{1}{2} . \frac{C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}} \frac{(1-\alpha_l)^2+1}{(1-\alpha_l)} \nabla\pi_s
\end{aligned}$$

Finally the geopotential term writes:

$$\begin{aligned}
(\nabla\phi)_l &= \nabla\phi_s + \sum_{k=l+1}^{L} \left\{ \frac{\delta_k}{1+\mathcal{P}_k} \nabla(RT)_k - \frac{R_k T_k \delta_k}{(1+\mathcal{P}_k)} \frac{\pi_k^*}{\pi_k} \nabla\hat{\mathcal{P}}_k \right\} \\
&+ \sum_{k=l+1}^{L} \left\{ \frac{R_k T_k}{1+\mathcal{P}_k} \left[\frac{\delta_k \mathcal{P}_k}{(1+\mathcal{P}_k)^2} \frac{1}{\delta\pi_k} \left(\frac{\delta_k}{\delta\pi_k} C_k + \delta B_k \right) - \frac{1}{2} \frac{C_k}{\pi_{\tilde{k}}\pi_{\tilde{k}-1}} \frac{(1-\alpha_k)^2+1}{(1-\alpha_k)} \right] \nabla\pi_s \right\} \\
&+ \frac{\alpha_l}{1+\mathcal{P}_l} \nabla(RT)_l - \frac{R_l T_l \alpha_l}{(1+\mathcal{P})} \frac{\pi_l^*}{\pi_l} \nabla\hat{\mathcal{P}}_l \\
&+ \frac{R_l T_l}{1+\mathcal{P}_l} \left[-\frac{1}{2} \frac{(1-\alpha_l)C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}} + \frac{\alpha_l}{1+\mathcal{P}_l} \frac{\mathcal{P}_l}{1+\mathcal{P}_l} \frac{1}{\delta\pi_l} \left(\frac{\delta_l}{\delta\pi_l} C_l + \delta B_l \right) \right] \nabla\pi_s \qquad (2.15)
\end{aligned}$$

Chapitre 3

Time Discretisation

(15/01/2001)

3.1 Introduction

The time discretisation is designed here in order to allow large time steps by mean of semi-implicit centered time scheme in the eulerian integration. The goal of this section in thus to give the expression of the temporal derivative operators, and to show how gravity and acoustic wave popagation can be controled by the semi-implicit technique. The basic explicit time stepping is performed with leap-frog scheme.

3.2 Semi-implicit Correction Formulation

The temporal discretisation is formulated as an explicit scheme with implicit correction. The advantage of this method is to allow the amount of implicitness of the scheme to be specified.
We now introduce a symbolic notation for the explanation of the principle of the semi-implicit scheme. Symbolically, the state of the model at time t can be represented by X^t (i.e. the whole grid point values or spectral coefficients of prognostic variables on the domain at time t), and the whole model evolution system applying to X by $\mathcal{M}$, which thus represents the system (1.45) – (1.49). The linear model (1.37) – (1.41) applying to X' can similarly be noted $\pounds$.
The original system (1.45) – (1.49) writes in continuous form:

$$\frac{\partial X}{\partial t} = \mathcal{M}.X$$

The linear associated system (1.37) – (1.41) writes:

$$\frac{\partial X'}{\partial t} = \pounds.X'$$

Examination of the linear system shows that, due to the structure of the basic state, it can be rewritten as:

$$\frac{\partial X}{\partial t} = \pounds.X$$

The semi-implicit time scheme then consists in discretising the evolution of the total model under the form:

$$\frac{\partial X}{\partial t} = \mathcal{M}.X + \beta\pounds\left(\overline{X} - X\right) \tag{3.1}$$

where $\overline{X}$ is a temporal average of X. Generally a simple centered average is chosen:

$$\overline{X} = \frac{X^+ + X^-}{2}$$

β is a parameter for activating the semi-implicit scheme. If $\beta = 0$, the scheme is explicit, and if $\beta = 1$, the scheme is semi-implicit. We can see that in the semi-implicit scheme, the linear evolution of the perturbation is temporally averaged. This has been shown by Tanguay *et al.*, 1990 to prevent the linear instability due to large speed of both acoustic and gravity propagating perturbations.
In certain cases an off-centered average (with a small ϵ decentering coefficient) can be chosen, to damp the most rapid oscillations:

$$\overline{X} = \frac{(1+\epsilon)X^+ + (1-\epsilon)X^-}{2}$$

The eulerian leap-frog time-discretised form of Eq. (3.1) is written as:

$$X^+ = X^- + 2\Delta t\ \mathcal{M}.X^0 + 2\beta\Delta t\ \pounds\left(\frac{X^+ + X^-}{2} - X^0\right) \tag{3.2}$$

Where X^+, X^0 and X^- respectively represent X at time $t+\Delta t$, t and $t-\Delta t$. Moving all the $t+\Delta t$ terms on the left hand side, it can be written as a linear sytem which has to be solved to get X^+:

$$\left(X^+ - \beta\Delta t\ \pounds X^+\right) = \underbrace{X^- + 2\Delta t\ \mathcal{M}.X^0}_{X_E^+} + \underbrace{\beta\Delta t\ \pounds\left(X^- - 2X^0\right)}_{\delta X_{ESI}^+}$$

In the right hand side, the first bracketed term is traditionally called "explicit guess" (subscript E) since it is the value that would be obtained with an explicit leap-frog scheme, and the second one, "explicit part of the semi-implicit correction" (subscript ESI). These two terms are computed and put together under the name $\widetilde{X^+}$.
The solution of the semi-implicit system then consists in inverting the linear system:

$$\left(X^+ - \beta\Delta t\ \pounds X^+\right) = \widetilde{X^+} \tag{3.3}$$

Of course, this problem is not treated as a whole but decomposed for each spectral coefficient and variable, under the form of vertical columns, as it will be depicted in the following sections.

3.3 The problem of Orography

The original linear system (1.37) – (1.41) was written for the variable perturbations X', and was valid in the presence of orography ϕ'_s. In the Eulerian context, the fact to apply $\pounds$ to $\left(\overline{X} - X\right)$ brings a further simplification in the momentum equation in the linear system: the surface geopotential term, stationnary in time by nature, naturally disappears:

$$-\frac{RT^*}{\pi^*}\Delta(\overline{\pi_s} - \pi_s) - \Delta(\overline{\phi_s} - \phi_s) \quad \longrightarrow \quad -\frac{RT^*}{\pi^*}\Delta(\overline{\pi_s} - \pi_s)$$

However this simplification of the linear system is only valid in the Eulerian context. In a semi-lagrangian context, the time-averaged geopotential $\overline{\phi_s}$ is not necessarily evaluated at the same position as the geopotential ϕ_s itself, and a special care must be taken to this problem, since there is no longer an automatic cancellation of the two terms.

A first solution is to evaluate $\overline{\phi_s}$ at the same points as ϕ_s (i.e. to spatially average the evaluation of this term) in order to readily eliminate the ϕ_s contribution as in the Eulerian case. This is what is done when using options NWLAG=2 or 3. Hence, the linear system and consequently the semi-implicit scheme are correct when using NWLAG=2 or 3,

Another solution would be to work with a modified variable $\pi_s + (\pi_s^*/RT^*)\phi_s$ everywhere in the model, instead of π_s, since this new variable is not contaminated by the orography in the linear system. Doing this leads to add in the RHS of all equations some Eulerian-type advective terms for orography, which can be treated explicitely. This possibility exists in ARPEGE and Aladin, and is applied when the "Eulerian treatment of orography is chosen (LEVGO=.T.). But unfortunately it is not implemented in a correct way: This transformation is done for the continuity and temperature equations, but curiously not for the momentum equation. This lack prevents the possibility to use the option NWLAG=1 when this "Eulerian treatment of the orography" is switched on, as it should be the case normally. By the way the option NWLAG=1 is not better than NWLAG=2 or 3, and is not convenient for two-time levels schemes, and so this lack is not really an handicap (moreover, the option NWLAG=1 will be soon removed form the code).

3.4 Transformations and Semi-implicit

At this stage, it is important to point out how the transformations occur in ARPEGE/Aladin. All computations are done on the cartesian ARPEGE/Aladin coordinate system, but they don't always apply on the same quantities. Spectral computations apply to the components of the "wind on the map" (u', v') (or D', ζ' pseudo divergence and vorticity on the map to be precise) and of the gradient on the map ∇', while grid point computations apply to the physical components of the wind (u, v) and of the gradient ∇. As a consequence, a special process takes place during the normal integration time loop, to switch alternatively from the (ζ', D') system to (u, v) one and *vice versa*.
To summarise schematically, let say that at the beginning of the time step, (ζ', D') are available in spectral coefficients. During subroutine ELINV, (u', v') are computed and stored, so that at the beginning of subroutine CPG, the two kinds of dynamical variables are available: (ζ', D', u', v'). In subroutine CPG, the variables are first transformed to geopgraphical variables (ζ, D, u, v), the explicit guess at $t + \Delta t$ and the explicit part of the semi-implicit correction are computed, but only for (u, v). At the end of subroutine CPG, these terms are transformed back to reduced variables, so CPG is left with: $(u'^{+}_{E}, \delta u'^{+}_{ESI}, v'^{+}_{E}, \delta v'^{+}_{ESI})$ in grid point values. In subroutine CPGLAG the wo terms for each variable are summed giving the quantities to be passed in input for solving the semi-implicit system: $(\widetilde{u'+}, \widetilde{v'+})$. In subroutine ELDIR, these terms are computed for the (ζ', D') system and stored, so that subroutine ESPC begins with $(\widetilde{u'+}, \widetilde{u'+}, \widetilde{\zeta'+}, \widetilde{D'+})$ available. In subroutine ESPC the semi implicit system is solved for (ζ', D'), giving these variables in spectral coefficients at time $t + \Delta t$.
(For further details, see ARPEGE note n° 22).
Since spectral computations are made with map variables and derivatives in the ARPEGE/Aladin grid coordinate system, the map factor has to be taken into account in the linear system. If the map factor is denoted $\mathbf{m}$, the horizontal pseudo divergence, gradient and laplacian write:

$$D' = D/\mathbf{m}^2 \tag{3.4}$$
$$\nabla' = \nabla/\mathbf{m} \tag{3.5}$$
$$\Delta' = \Delta/\mathbf{m}^2 \tag{3.6}$$

Finally, this system is linearised with respect to $\mathbf{m}$ in order to get horizontally constant coefficients for the variables. The map factor $\mathbf{m}$ is thus replaced by its maximum value $\overline{\mathbf{m}}$. This simplification appears to be legitimate for limited area models in which $\mathbf{m}$ remains close to unity. In ARPEGE global model, this simplification can't be done any more, resulting in an extra penta-diagonal matrix inversion.
The linear system (1.37) – (1.41) then writes:

$$
\begin{aligned}
\frac{\partial D'}{\partial t} &= -R\mathcal{G}^* \Delta' T' + gH\mathcal{G}^* \Delta' \hat{\mathcal{P}} - RT^* \Delta' \hat{\mathcal{P}} - \frac{RT^*}{\pi_s^*} \Delta' \pi_s' - \Delta' \phi_s' \\
\frac{\partial T'}{\partial t} &= -\frac{RT^*}{C_v} (\overline{\mathbf{m}}^2 D' + \hat{d}) \\
\frac{\partial \pi_s'}{\partial t} &= -\pi_s^* \mathcal{N}^* \overline{\mathbf{m}}^2 D' \\
\frac{\partial \hat{\mathcal{P}}}{\partial t} &= \mathcal{S}^* \overline{\mathbf{m}}^2 D' - \frac{C_p}{C_v} (\overline{\mathbf{m}}^2 D' + \hat{d}) \\
\frac{\partial \hat{d}}{\partial t} &= -\frac{g}{H} \mathcal{L}^* \hat{\mathcal{P}}
\end{aligned} \tag{3.7}
$$

It can be seen from the above system, that it is now purely local (i.e. the coefficients of the linear system are horizontally constant), which implies separability. The variables are now considered as being column vectors of vertical values from level 1 to L for a given spectral component (m, n). Similarly, the linear operators are considered as being matrices applying to these vectors.

3.5 Implementation of the Semi-implicit

The system (3.3) is now cast in spectral space for each individual variable column at spectral location (m, n), starting from (3.2), and ignoring the geopotential terms as explained in section 3.3. The vertical column vector corresponding to a variable Q is noted $\underline{Q}$ (and similarly $\underline{\pi_s} = \pi_s.(1, 1, ...1)$).

$$
\begin{aligned}
\underline{D'}^+ - \beta\Delta t \, \Delta' \left[RT^* (\mathbf{G}^* - 1) \underline{\hat{\mathcal{P}}}^+ - R\mathbf{G}^* \underline{T}^+ - \frac{RT^*}{\pi_s^*} \underline{\pi_s}^+ \right] &= \underline{\widetilde{D'+}} \\
\underline{\hat{d}}^+ - \beta\Delta t \left(-\frac{g}{H} \mathbf{L}^* \underline{\hat{\mathcal{P}}}^+ \right) &= \underline{\widetilde{\hat{d}+}} \\
\underline{\hat{\mathcal{P}}}^+ - \beta\Delta t \left[\left(\mathbf{S}^* - \frac{C_p}{C_v} \right) \overline{\mathbf{m}}^2 \underline{D'}^+ - \frac{C_p}{C_v} \underline{\hat{d}}^+ \right] &= \underline{\widetilde{\hat{\mathcal{P}}+}} \\
\underline{T}^+ - \beta\Delta t \left(-\frac{RT^*}{C_v} \overline{\mathbf{m}}^2 \underline{D'}^+ - \frac{RT^*}{C_v} \underline{\hat{d}}^+ \right) &= \underline{\widetilde{T+}} \\
\underline{\pi_s}^+ - \beta\Delta t \left(-\pi_s^* \mathbf{N}^* \overline{\mathbf{m}}^2 \underline{D'}^+ \right) &= \underline{\widetilde{\pi_s^+}}
\end{aligned}
$$

with:

$$
\begin{aligned}
\underline{\widetilde{D'+}} &= \frac{1}{\mathbf{m}^2} \underline{D_E^+} + \frac{1}{\mathbf{m}^2} \underline{\delta D_{ESI}^+} \\
\underline{\widetilde{\hat{d}+}} &= \underline{\hat{d}_E^+} + \underline{\delta \hat{d}_{ESI}^+} \\
\underline{\widetilde{\hat{\mathcal{P}}+}} &= \underline{\hat{\mathcal{P}}_E^+} + \underline{\delta \hat{\mathcal{P}}_{ESI}^+} \\
\underline{\widetilde{T+}} &= \underline{T_E^+} + \underline{\delta T_{ESI}^+} \\
\underline{\widetilde{\pi_s^+}} &= \underline{\pi_{sE}^+} + \underline{\delta \pi_{s\,ESI}^+}
\end{aligned}
$$

and:

$$
\begin{aligned}
\underline{\delta D_{ESI}^+} &= \beta\Delta t \, \Delta \left[RT^* (\mathbf{G}^* - 1) \left(\underline{\hat{\mathcal{P}}}^- - 2\underline{\hat{\mathcal{P}}}^0 \right) - R\mathbf{G}^* \left(\underline{T}^- - 2\underline{T}^0 \right) - \frac{RT^*}{\pi_s^*} \left(\underline{\pi_s}^- - 2\underline{\pi_s}^0 \right) \right] \\
\underline{\delta \hat{d}_{ESI}^+} &= \beta\Delta t \left(-\frac{g}{H} \mathbf{L}^* \right) \left(\underline{\hat{\mathcal{P}}}^- - 2\underline{\hat{\mathcal{P}}}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\delta\hat{\mathcal{P}}^{+}_{ESI}} &= \beta\Delta t\left[\left(\mathbf{S}^{*}-\frac{C_p}{C_v}\right)\frac{\overline{\mathbf{m}}^2}{\mathbf{m}^2}\left(\underline{D}^{-}-2\underline{D}^{0}\right)-\frac{C_p}{C_v}\left(\underline{\hat{d}}^{-}-2\underline{\hat{d}}^{0}\right)\right]\\
\underline{\delta T^{+}_{ESI}} &= \beta\Delta t\left[-\frac{RT^{*}}{C_v}\frac{\overline{\mathbf{m}}^2}{\mathbf{m}^2}\left(\underline{D}^{-}-2\underline{D}^{0}\right)-\frac{RT^{*}}{C_v}\left(\underline{\hat{d}}^{-}-2\underline{\hat{d}}^{0}\right)\right]\\
\underline{\delta\pi^{+}_{s\,ESI}} &= \beta\Delta t\left[-\pi_s^{*}\mathbf{N}^{*}\frac{\overline{\mathbf{m}}^2}{\mathbf{m}^2}\left(\underline{D}^{-}-2\underline{D}^{0}\right)\right]
\end{aligned}
$$

Note that the computation of tilded quantities being made in grid point space, they use only physical quantities. Moreover, the computation of $\widetilde{\underline{D'+}}$ is in fact decomposed in the code, exactely as in the hydrostatic version, by first computing $\widetilde{\underline{u'+}}$ and $\widetilde{\underline{v'+}}$ in grid point space and then combining them during spectral transforms as mentionned above. We won't go into more details about this subject since it is not a non-hydrostatic specificity.

3.5.1 Elimination of variables

The elimination process is similar to the derivation of the structure equation of the linear system. The $\underline{\hat{d}}^{+}$ and $\underline{D'}^{+}$ variable are first kept:

$$
\begin{aligned}
\underline{\hat{d}}^{+} &= \underline{\hat{d}}^{\star}+\beta^2\Delta t^2\left[\left(-\frac{g}{H}\mathbf{L}^{*}\right)\left(\mathbf{S}^{*}-\frac{C_p}{C_v}\right)\overline{\mathbf{m}}^2\underline{D'}^{+}+\left(-\frac{g}{H}\mathbf{L}^{*}\right)\left(-\frac{C_p}{C_v}\right)\underline{\hat{d}}^{+}\right]\\
&= \underline{\hat{d}}^{\star}+\beta^2\Delta t^2\left[\frac{\mathbf{L}^{*}}{H^2}\left(-RT^{*}\mathbf{S}^{*}+c^2\right)\overline{\mathbf{m}}^2\underline{D'}^{+}+c^2\frac{\mathbf{L}^{*}}{H^2}\underline{\hat{d}}^{+}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{D'}^{+} = \underline{D}^{\star}+\beta^2\Delta t^2\Delta'\Bigg\{&\left[RT^{*}\left(\mathbf{G}^{*}-1\right)\left(\mathbf{S}^{*}-\frac{C_p}{C_v}\right)+\left(-R\mathbf{G}^{*}\right)\left(-\frac{RT^{*}}{C_v}\right)+\right.\\
&\left.\left(-\frac{RT^{*}}{\pi_s^{*}}\right)\left(-\pi_s^{*}\mathbf{N}^{*}\right)\right]\overline{\mathbf{m}}^2\underline{D'}^{+}+\left[RT^{*}\left(\mathbf{G}^{*}-1\right)\left(-\frac{C_p}{C_v}\right)+\left(-R\mathbf{G}^{*}\right)\left(-\frac{RT^{*}}{C_v}\right)\right]\underline{\hat{d}}^{+}\Bigg\}
\end{aligned}
$$

which yields, by virtue of (2.1):

$$
\underline{D'}^{+}=\underline{D}^{\star}+\beta^2\Delta t^2\Delta'\left[c^2\overline{\mathbf{m}}^2\underline{D'}^{+}+\left(-RT^{*}\mathbf{G}^{*}+c^2\right)\underline{\hat{d}}^{+}\right]
$$

with following explicit terms:

$$
\begin{aligned}
\underline{\hat{d}}^{\star} &= \widetilde{\underline{\hat{d}+}}+\beta\Delta t\left[\left(-\frac{g}{H}\mathbf{L}^{*}\right)\widetilde{\underline{\hat{\mathcal{P}}+}}\right]\\
\underline{D}^{\star} &= \widetilde{\underline{D'+}}+\beta\Delta t\Delta'\left[RT^{*}\left(\mathbf{G}^{*}-1\right)\widetilde{\underline{\hat{\mathcal{P}}+}}+\left(-R\mathbf{G}^{*}\right)\widetilde{\underline{T+}}+\left(-\frac{RT^{*}}{\pi_s^{*}}\right)\widetilde{\underline{\pi_s^{+}}}\right]
\end{aligned}
$$

The linear system can be rearranged:

$$
\begin{aligned}
\left(1-\beta^2\Delta t^2c^2\frac{\mathbf{L}^{*}}{H^2}\right)\underline{\hat{d}}^{+} &= \underline{\hat{d}}^{\star}+\beta^2\Delta t^2\frac{\mathbf{L}^{*}}{H^2}\left(-RT^{*}\mathbf{S}^{*}+c^2\right)\overline{\mathbf{m}}^2\underline{D'}^{+}\\
\left(1-\beta^2\Delta t^2c^2\overline{\mathbf{m}}^2\Delta'\right)\underline{D'}^{+} &= \underline{D}^{\star}+\beta^2\Delta t^2\Delta'\left(-RT^{*}\mathbf{G}^{*}+c^2\right)\underline{\hat{d}}^{+}
\end{aligned}
$$

Finishing the elimination with $\underline{\hat{d}}^{+}$ is done by combining the two latter equations and using the commutativity between horizontal and vertical operators:

$$
\begin{aligned}
&\left(1-\beta^2\Delta t^2 c^2\overline{\mathbf{m}}^2\Delta'\right)\left(1-\beta^2\Delta t^2 c^2\frac{\mathbf{L}^*}{H^2}\right)\underline{\hat{d}}^+ \\
&= \left(1-\beta^2\Delta t^2 c^2\overline{\mathbf{m}}^2\Delta'\right)\left[\underline{\hat{d}}^\star+\beta^2\Delta t^2\frac{\mathbf{L}^*}{H^2}\left(-RT^*\mathbf{S}^*+c^2\right)\overline{\mathbf{m}}^2\underline{D'}^+\right] \\
&= \left(1-\beta^2\Delta t^2 c^2\overline{\mathbf{m}}^2\Delta'\right)\underline{\hat{d}}^\star+\beta^2\Delta t^2\frac{\mathbf{L}^*}{H^2}\left(-RT^*\mathbf{S}^*+c^2\right)\left(1-\beta^2\overline{\mathbf{m}}^2\Delta t^2 c^2\Delta'\right)\overline{\mathbf{m}}^2\underline{D'}^+ \\
&= \left(1-\beta^2\Delta t^2 c^2\overline{\mathbf{m}}^2\Delta'\right)\underline{\hat{d}}^\star+\beta^2\Delta t^2\frac{\mathbf{L}^*}{H^2}\left(-RT^*\mathbf{S}^*+c^2\right)\overline{\mathbf{m}}^2\left[\underline{D}^\star+\beta^2\Delta t^2\Delta'\left(-RT^*\mathbf{G}^*+c^2\right)\underline{\hat{d}}^+\right]
\end{aligned}
$$

By using the second discretisation constraint (2.3), the expression for $\underline{\hat{d}}^+$ takes the final form:

$$
\mathbf{B}\underline{\hat{d}}^+ = \left[1-\beta^2\Delta t^2 c^2\left(\overline{\mathbf{m}}^2\Delta'+\frac{\mathbf{L}^*}{H^2}\right)-\beta^4\Delta t^4 N^2 c^2\overline{\mathbf{m}}^2\Delta'\mathbf{T}^*\right]\underline{\hat{d}}^+ = \underline{\hat{d}}^{\star\star} \tag{3.8}
$$

with:

$$
\underline{\hat{d}}^{\star\star} = \left(1-\beta^2\Delta t^2 c^2\overline{\mathbf{m}}^2\Delta'\right)\underline{\hat{d}}^\star+\beta^2\Delta t^2\frac{\mathbf{L}^*}{H^2}\left(-RT^*\mathbf{S}^*+c^2\right)\overline{\mathbf{m}}^2\underline{D}^\star \tag{3.9}
$$

Finally:

$$
\underline{\hat{d}}^+ = \mathbf{B}^{-1}.\underline{\hat{d}}^{\star\star} \tag{3.10}
$$

It is now clear that the elimination of the semi-implicit system fundamentally results in the inversion of a tridiagonal matrix $\mathbf{B}$. This has to be done for each spectral component (m,n). However, this inversion can be done once only at the beginning of the integration since the $\mathbf{B}$ matrix is independant of time. Practically, all the matrices $\mathbf{B}_m^n$ are computed, inverted and then stored in memory for use during the integration.
The locality of the discretised scheme has been preserved, since all involved operators are tridiagonal. This is true for the LHS as already stated in the vertical discretisation chapter. But this is also true for the RHS of the Helmholtz equation (3.10). The RHS contains the operators $\mathbf{I}^*$, $\mathbf{L}^*$,

3.5.2 Determination of variables at $t+\Delta t$

Once $\underline{\hat{d}}^+$ has been determined, the computation of the other prognostic variables at $t+\Delta t$ is straightforward. First, the horizontal divergence is obtained by inversion of a diagonal matrix, which can thus be solved scalarly:

$$
\underline{D'}^+ = \left(1-\beta^2\Delta t^2 c^2\overline{\mathbf{m}}^2\Delta'\right)^{-1}\left[\underline{D}^\star+\beta^2\Delta t^2\Delta'\left(-RT^*\mathbf{G}^*+c^2\right)\underline{\hat{d}}^+\right] \tag{3.11}
$$

The other variables are directly obtained from the original system:

$$
\begin{aligned}
\underline{\hat{\mathcal{P}}}^+ &= \underline{\widetilde{\hat{\mathcal{P}}+}}+\beta\Delta t\left[\left(\mathbf{S}^*-\frac{C_p}{C_v}\right)\overline{\mathbf{m}}^2\underline{D'}^+-\frac{C_p}{C_v}\underline{\hat{d}}^+\right] \\
\underline{T}^+ &= \underline{\widetilde{T+}}+\beta\Delta t\left(-\frac{RT^*}{C_v}\overline{\mathbf{m}}^2\underline{D'}^+-\frac{RT^*}{C_v}\underline{\hat{d}}^+\right) \\
\underline{\pi_s}^+ &= \underline{\widetilde{\pi_s^+}}+\beta\Delta t\left(-\pi_s^*\mathbf{N}^*\overline{\mathbf{m}}^2\underline{D'}^+\right)
\end{aligned}
$$

3.6 Iteration of a Part of Semi-Implicit

3.6.1 Introduction

It has been found in practical experimentation that the the non-linear part of the 3-dimensional divergence D_3 (Eq. 1.53) was responsible of an instability due to the insufficient control of fast waves which made the computational mode growing continuously. This kind of instability is mentionned in Ikawa (1988). It can be treated in a similar way as in Côté and Staniforth (1988) by performing a temporal average not only on the linear part of D_3 (as done by the semi-implicit scheme) but on the whole D_3 quantity. This can be done through an iterative process in which the corrected (i.e. time-averaged) non-linear variables are used to re-enter the semi-implicit scheme. Of course this process must be shown to be economically possible, i.e. only a small part of the total model time-step computations must be iterated, unless the overcost of this correction would become too costly.

The problematic non-linear part of D_3 will be noted Y:

$$D_3 = (D' + \hat{d}) + Y \tag{3.12}$$

Thus:

$$Y = \frac{1}{m}\frac{p}{RT}\left(\frac{\partial \mathbf{V}}{\partial \eta}\right).\nabla\phi + \hat{d}\left(\frac{\partial \phi^*}{\partial \phi} - 1\right) = -\left(\frac{\partial \mathbf{V}}{\partial \phi}\right).\nabla\phi + \hat{d}\left(\frac{\partial \phi^*}{\partial \phi} - 1\right)$$

First let us define the notations for the iteration scheme. The iteration index is noted $i = (0, 1, ...)$. In the equation (3.3), i-eth iterated values of X^+ (resp. $\widetilde{X+}$) are noted $X^{+(i)}$ (resp. $\widetilde{X^{+(i)}}$). Thus the equation (3.3) can be rewritten as:

$$\left(X^{+(i)} - \beta\Delta t \; \pounds X^{+(i)}\right) = \widetilde{X^{+(i)}} \tag{3.13}$$

and $\widetilde{X^{+(i+1)}}$ will be computed from $\widetilde{X^{+(0)}}$ and $X^{+(i)}$ as explained in the following section.

The principle of the process will be as following:

i) Compute the RHS $\widetilde{X^{+(0)}}$ of the SI scheme in GP space

ii) From $\widetilde{X^{+(0)}}$, compute the "normal" semi-implicitly averaged $X^{+(0)}$ values by spectrally inverting the Helmholtz equation.

iii) back in GP space, diagnose the cross-term $Y^{+(0)}$ from the obtained $X^{+(0)}$ values

iv) Use $Y^{+(0)}$ to temporally average Y in the RHS of the semi-implicit system at the places where D_3 originally was appearing (i.e. in T and $\hat{\mathcal{P}}$ equations) thus resulting in a new RHS $\widetilde{X^{+(1)}}$

v) iterate the three sub-processes (ii-iv) while incrementing the iteration index in parenthesis

It should be noted that the sub-processes (ii) is made in spectral space whereas (i, iii, iv) one are done in grid-point space. The iteration process thus requires the computation of a certain amount of transformations

.

It should also be noted that the term $\partial\mathbf{V}/\partial\phi$, which must be evaluated on levels l has no natural discretizated expression. Practically, this term is discretized at level l using wind information from levels $(l-1,\, l,\, l+1)$ with some discretization weights ϵ defined in Eq. (2.13).

3.6.2 Implementation

From the beginning, and as usual we first compute $\widetilde{X^{+(0)}}$ from every variables at times past and present. Then the inversion of the SI operator leads to $X^{+(0)}$ which is the normal values in the case of no iteration. Then the value of $Y^{+(0)}$ is diagnosed from the set of previous ones:

$$Y^{+(0)} = -\left(\frac{\partial \mathbf{V}^{+(0)}}{\partial \phi^{+(0)}}\right).\nabla\phi^{+(0)} + \hat{d}^{+(0)}\left(\frac{\partial \phi^*}{\partial \phi^{+(0)}} - 1\right) \tag{3.14}$$

N.B: In practice, for the computation of $Y^{+(0)}$ from the above equation, some coefficients can be taken as non-iterated at time 0. These are the "strangers" defined and explained in next section.

To compute the next iterated $\widetilde{X^{+(0)}}$ values, only the part of the system equation which involve D_3 is used (and only the leading term for $\hat{\mathcal{P}}$ equation):

$$\begin{aligned}
\frac{dT}{dt} &= -\frac{RT}{C_v} D_3 + ... &(3.15)\\
\frac{d\hat{\mathcal{P}}}{dt} &= -\frac{C_p}{C_v}\frac{\pi}{\pi^*} D_3 + ... &(3.16)
\end{aligned}$$

Thus we define:

$$\begin{aligned}
\widetilde{T^{+(1)}} &= \widetilde{T^{+(0)}} - 2\beta\Delta t \frac{RT^*}{C_v}\left(\frac{Y^{+(0)} + Y^-}{2} - Y^0\right) &(3.17)\\
\widetilde{\hat{\mathcal{P}}^{+(1)}} &= \widetilde{\hat{\mathcal{P}}^{+(0)}} - 2\beta\Delta t \frac{C_p}{C_v}\left(\frac{Y^{+(0)} + Y^-}{2} - Y^0\right) &(3.18)\\
\widetilde{D^{+(1)}} &= \widetilde{D^{+(0)}} &(3.19)\\
\widetilde{\hat{d}^{+(1)}} &= \hat{d}^{+(0)} &(3.20)
\end{aligned}$$

then, this new RHS $\widetilde{X^{+(1)}}$ is used as an entry for the SI operator:

$$\left(X^{+(1)} - \beta\Delta t \ \pounds X^{+(1)}\right) = \widetilde{X^{+(1)}} \qquad (3.21)$$

For a given further iteration (i) = 2,3,..., assuming the $(i-1)$-th iteration has been completed (i.e. that $X^{+(i-1)}$ has been computed, the value of $Y^{+(i-1)}$ is diagnosed from $X^{+(i-1)}$:

$$Y^{+(i-1)} = -\left(\frac{\partial \mathbf{V}^{+(i-1)}}{\partial \phi^{+(i-1)}}\right).\nabla\phi^{+(i-1)} + \hat{d}^{+(i-1)}\left(\frac{\partial \phi^*}{\partial \phi^{+(i-1)}} - 1\right) \qquad (3.22)$$

N.B: As for $i = 1$ above, in practice for the computation of $Y^{+(i-1)}$ from the above equation, some coefficients can be taken as non-iterated at time 0. These are the "strangers" defined and explained in next section.

Then we define:

$$\begin{aligned}
\widetilde{T^{+(i)}} &= \widetilde{T^{+(0)}} - 2\beta\Delta t \frac{RT^*}{C_v}\left(\frac{Y^{+(i-1)} + Y^-}{2} - Y^0\right) &(3.23)\\
\widetilde{\hat{\mathcal{P}}^{+(i)}} &= \widetilde{\hat{\mathcal{P}}^{+(0)}} - 2\beta\Delta t \frac{C_p}{C_v}\left(\frac{Y^{+(i-1)} + Y^-}{2} - Y^0\right) &(3.24)\\
\widetilde{D^{+(i)}} &= \widetilde{D^{+(0)}} &(3.25)\\
\widetilde{\hat{d}^{+(i)}} &= \hat{d}^{+(0)} &(3.26)
\end{aligned}$$

then, this new RHS $\widetilde{X^{+(i)}}$ is used as an entry for the SI operator:

$$\left(X^{+(i)} - \beta\Delta t \ \pounds X^{+(i)}\right) = \widetilde{X^{+(i)}} \qquad (3.27)$$

If we look carefully (on the T equation for instance) we see that after one iteration, the SI equation to be inverted is:

$$
\begin{aligned}
T^{+(1)} &= T^{-} + 2\Delta t\left(\frac{-RT}{C_v}D_3^0\right) + \frac{Q}{C_v} + ... \\
&+ 2\beta\Delta t\left(\frac{-RT^*}{C_v}\right)\left[\left(\frac{D^{+(1)}+D^{-}}{2} - D^0\right) + \left(\frac{\hat{d}^{+(1)}+\hat{d}^{-}}{2} - \hat{d}^0\right) + \left(\frac{Y^{+(0)}+Y^{-}}{2} - Y^0\right)\right]
\end{aligned}
$$

showing that the whole (linear and non-linear) part of the D_3 term is time-averaged in the process. The approximation due to the iterative process lies in the fact that in the last parenthesis $Y^{+(0)}$ appears instead of $Y^{+(1)}$ as it would be for an exact process. Nevertheless, this way of doing is justified *a posteriori* if the iterative process converges rapidly. This has been shown to be the case: the differences $(Y^{+(i+1)} - Y^{+(i)})$ decrease very fast (e.g. one order of magnitude at each iteration) for typical cases. This has allowed us to estimate that practically, one iteration only is enough for stability and accuracy of the process.
Thus in the following, we note:

$$X^{+} = X^{+(1)} \tag{3.28}$$

3.6.3 Computational cost aspects

in term of cost, this process can be made not very expensive. Several aspects have to be considered:

- Minimization grid-point computations for $i > 0$.
- Minimization the number of fields to be transformed to and from spectral space.
- Minimization the spectral computations.

It has been shown that not every terms in Eq. (3.14) absolutely requires to be computed from the values obtained after the previous iteration is completed. In fact, $\nabla\phi^0$ can be use with no detrimental effect instead of $\phi^{+(0)}$. Similarly, the discretization weights for evaluating wind shear, and reduced gaz constant $R/(1+\mathcal{P})$ can be kept as old values without deterioration of the scheme. Of course $2Y^0 - Y^-$ can be also stored once at the beginning of the iterative process. This allows to save the computation time which would be necessary for their computation. But on another hand, requires additional space for keeping the values of these grid-point fields. These five fields are traditionnally called as "strangers" in the code since they do not participate to the normal evolution, but remains unchanged during the whole iteration process.

$$\text{[Strangers]} = 2Y^0 - Y^-,\ R'^0,\ \nabla\phi^0,\ \epsilon^0$$

When the normal computation of $X^{+(i)}$ is achieved in spectral space, only those quantities which are necessary to compute $Y^{+(i)}$ from Eq. (3.14) in grid-point space should be (inverse Fourier) transformed, thus minimizing the cost of transforms spent for this iterative process. However, this cost optimization is not yet implemented in the code.
By construction, there is no special optimization to be done for spectral computations: the inversion of SI operator must simply be (re)done with the new iterated RHS.

3.7 The Problem of the Additional Tridiagonal Matrix $\mathbf{T}^*$

3.7.1 Introduction

It has been seen in the vertical discretisation chapter that the additional $\mathbf{T}^*$ matrix appearing in the LHS of the SI equation did not tends towards identity as the number of levels increases: on contrary it tends towards a tridiagonal regular averaging operator:

$$\mathbf{T}^* \quad \longrightarrow \quad \mathbf{I}^* + \frac{1}{4}\mathbf{D}_2^*$$

We will evaluate here how this modification is important and what kind of modification can be expected, compared to the continuous case or to the case $\mathbf{T}^* = \mathbf{I}^*$.

First of all it should be noted that the second constraint arises only when one tries to perform the elimination of variables in the linear system, to determine the properties of the allowed "fast" stationnary modes. The other "slow" modes (Rossby modes, "Lagrangian" perturbations embeded in the flow...) are not concerned by this approximation: all the description of the non-wave processes is thus treated accurately in this sense. Of course, the vertical operators are not accurately defined at the edges of the domain (for instance $\mathbf{L}^*$ applied to a constant field is not zero at the bottom of the domain), but this limitation is a different problem which cannot be avoided in grid point representation.

3.7.2 Impact on the time-continuous wave propagation

The time-continuous space-discretised structure equation (1.44) already contains the modification brought by the approximation. The wave propagation is thus affected even in the time-continuous system. To simplify the discussion, we assume here that elastic and gravity modes can be considered as pure modes instead of mixed modes. The pure elastic waves will not be affected by the approximation since these modes do not involve the last term of the structure equation's LHS. On the other hand, the pure gravity waves propagation is likely to be affected. For a given gravity mode of frequency $(\omega/2\pi)$ the space-discretised structure equation will write, after grouping all vertical operators:

$$-\omega^2 \left[\Delta' + \frac{\mathbf{L}^*}{H^2} \left(\mathbf{I}^* - \frac{N^2 H^2}{\omega^2} \mathbf{Q}^* \Delta' \right) \right] + N^2 \Delta' = 0$$

We see that the propagation is not significantly modified if the matrix $(N^2H^2/\omega^2)\mathbf{Q}^*\Delta'$ remain small. For a thin enough discretisation, the magnitude of $\mathbf{Q}^*$ is close from $\delta_l^2/4 = (\delta z_l/H)^2/4$ where δz_l is the physical depth of the level. Hence, for a given horizontal wave number k, the condition writes:

$$\frac{N^2 k^2 H^2 \delta^2}{4\omega^2} \ll 1$$

Finally, applying the original dispersion equation for the pure NH gravity waves $\omega^2\sqrt{k^2+n^2} = k^2N^2$ (where n is the vertical wave number), the condition for not modifying the wave propagation becomes:

$$\delta^2 H^2 \sqrt{k^2+n^2} = \delta z^2 \sqrt{k^2+n^2} \ll 4 \tag{3.29}$$

Since usually $k \ll n$, the propagation of waves will not be affected if $n.\delta z \ll 2$.

For other modes (the shortest ones) the approximation of the second constraint modifies the spatio-temporal structure. However, determining the stationnary modes for a given k is not trivial since it is no more a classical eigenmode problem. The problem now consists in finding modes (ω, d) such as the following equation is verified:

$$\left(\omega^2 - N^2\right) k^2 - \left[\frac{\mathbf{L}^*}{H^2} \left(\omega^2 + N^2 H^2 k^2 \mathbf{Q}^* \right) \right] = 0$$

This kind of problem cannot be easily solved in the general case. The only way to come back to a classical eigenmode problem is to have a scalar matrix for $\mathbf{Q}^*$. This can be achieved by assuming a constant vertical resolution.

Analysis for constant vertical resolution

If the vertical resolution is constant, then $H\delta_l = \delta z_l$ is a constant and the $\mathbf{Q}^*$ matrix is close of a scalar matrix (it is scalar except at the top where it is zero). Here we will assume that the vertical resolution is constant and that $\mathbf{Q}^*$ becomes scalar, hence with the value $\delta^2/4 = (\delta z/2H)^2$. In this case, the problem becomes an eigenmodes problem and the dispersion equation is:

$$\left(\omega^2 - N^2\right) k^2 + n^2 \left(\omega^2 + N^2 H^2 k^2 \delta^2/4\right) = 0$$

or:

$$\omega^2 = \frac{N^2 k^2}{k^2 + n^2} \left(1 - \frac{n^2 H^2 \delta^2}{4}\right)$$

We recognize the classical pure gravity wave equation, but modified by the factor between parenthesis in the RHS. This factor can become negative for very short vertical wavelengths. For these modes, the frequency becomes imaginary, and the amplitude is increasing.
The approximation due to the fact that $\mathbf{Q}^*$ is not really a scalar can be evaluated by computing the difference $\mathbf{L}^*.\mathbf{Q}^* - (\delta/4)\mathbf{L}^*$ applied to the shortest eigenvectors of $\mathbf{L}^*$. The difference is found to be negligible in comparison of the two vectors , thus justifying the assumption.
For thin vertical discretisations this appears to be not a real problem since very short vertical modes practically don't exist in the atmosphere (most of the energy of gravity waves, those forced by dynamical effects, is around the vertical wavelength ≈ 2km corresponding to a relative wind of 10-20 m/s). For the short vertical waves, the propagation is affected in the sense that the vertical structure is given by the eigenmodes of the new approximated matrix:

$$\frac{\mathbf{L}^*_{\text{app}}}{H^2} = \frac{\mathbf{L}^*}{H^2} \left(\mathbf{I}^* - \frac{N^2 H^2}{\omega^2} \mathbf{Q}^* \Delta'\right) \tag{3.30}$$

compared to the exact case, the vertical structure corresponding to a given mode (k, ω) is slightly different from the exact case, and the vertical structure now depends on k (i.e. there are no more separable vertical modes for these waves).
In the same manner, the propagation velocities are modified. The $\mathbf{T}^*$ has positive eigenvalues λ, but smaller than 1 (especially for short vertical modes whose eigenvalues tends to zero). For a given k, the matrix in parenthesis in (3.30) has all its positive eigenvalues λ^2 in the interval:

$$\lambda^2 \in \left[\left(1 - N^2 H^2 k^2 \delta_0^2/\omega^2\right), 1\right]$$

where δ_0^2 is the largest of δ_l^2 values. The new continuous dispersion equation will have the form:

$$\omega_c^2 = \frac{N^2 k^2 \lambda^2}{k^2 + n^2} \tag{3.31}$$

The continuous phase and group velocity are then reduced due to the approximation. Since no imaginary part appears in the frequency, the modification will not bring any damping/amplifying effect.

3.7.3 Impact on the explicit model integrations

The non-linear model is designed with formally the same vertical operators as the linear one. This means that the full explicit model itself will be affected by the fact that the second constraint is not verified in the discrete case for the non-linear operators. Basically, the vertical Laplacian and the vertical integral operators are not "balanced" to fulfil this constraint.
For the fast modes, the propagation is affected in the same way as in the time-continuous case. Only the short vertical gravity modes are affected. The modification basically consists in a modification of the frequency and

by the fact that for these short waves, the vertical structure becomes linked to the horizontal one. Since the continuous phase speed is reduced, the stability criterion of the time scheme is not more stringent.

3.7.4 Impact on the semi-implicit model integrations

In a similar way as in the continuous case, the vertical operators can be grouped in the Helmholtz SI equation:

$$\left[1-\beta^2\Delta t^2 c^2\left(\overline{\mathbf{m}}^2\Delta' + \frac{\mathbf{L}^*}{H^2}\left(1+\beta^2\Delta t^2 N^2 H^2\overline{\mathbf{m}}^2\Delta'\mathbf{Q}^*\right)\right) - \beta^4\Delta t^4 N^2 c^2\overline{\mathbf{m}}^2\Delta'\right]\underline{\hat{d}}^{+} = \underline{\hat{d}}^{\star\star} \tag{3.32}$$

As in the continuous case, the modification leads to a vertical operator which has a dependancy to horizontal modes, and hence, there are no more separable vertical modes. However, we will see that for the large majority of modes, the propagation is not significantly modified.
For given Δt and k, the stationnary modes are the eigenmodes of the approximated vertical Laplacian operator:

$$\frac{\mathbf{L}^*_{\text{app}}}{H^2} = \frac{\mathbf{L}^*}{H^2}\left(\mathbf{I}^*\beta^2\Delta t^2 N^2 H^2\overline{\mathbf{m}}^2 k^2\mathbf{Q}^*\right)$$

$\mathbf{Q}^*$ is a diagonal matrix whose magnitude tends towards $\delta_l^2/4$ when the vertical resolution increases. Its impact will therefore become negligible if $\Delta t N H k\delta_l/2 \ll 1$ for every levels. Since $H\delta_l = \delta z_l$ is the physical depth of the layer, and $k > \pi/\Delta x$, the condition writes:

$$\frac{\pi}{2}N\frac{\Delta t}{\Delta x}\delta z_l \ll 1$$

For typical NWP applications, this gives $\delta z_l \ll 11000$m. This condition is usually fulfilled near the bottom of the domain but not necessarily near the top. The modifications brought by this approximations are thus likely to occur only near the top (and only for certain wave numbers).
The SI dispersion is always given by:

$$\omega_{\text{SI}}\Delta t = \arctan\left(\omega_c\Delta t\right)$$

where ω_c is given by (3.31). Therefore, the SI linear system still is unconditionally stable. For those modes which are modified by the approximation, the modification still consists in decreasing the propagation velocities by a factor linked to the concerned eigenvalue of $\mathbf{T}^*$. Nevertheless, the modification appears to be of the same kind as the one brought by the SI treatment, i.e. it consists in adding an extra dispersivity to the natural continuous one.

3.7.5 Summary

The impossibility to fulfil the second constraint leads to a modification of the propagation of the gravity waves. Altough the discrete system does not converge towards the continuous one as the vertical resolution increases, this can be considered as not important for the model for the following reasons:

- The stability of the model is not endangered
- The modification does not involve the slow part of the flow but only the fast modes.
- The large majority of the waves are not significantly affected
- For those waves which are affected, the modification is not dramatic, it only consists in an additional dispersivity.
- The waves which play a meteorological role are not significantly affected by the modification.
- In any case, it should never be forgotten that the overall meteorological importance of fast wave always remains weak in a numerical NWP model

3.8 Time Filter

A time filter is used to damp fast time oscillations of variables due to the decorelation of consecutive time levels in the leap frog scheme. The formulation is the same as in ARPEGE: The filter can be uncentered, as shown in the expression for filtered X variable (noted $\overline{X}$):

$$\overline{X}^t = \varepsilon_1 \overline{X}^{t-1} + (1 - \varepsilon_1 - \varepsilon_2) X^t + \varepsilon_2 X^{t+1}$$

This time filter is applied in grid point space, and computations are splitted into two steps. The first part of $\overline{X}^t$, involving $\overline{X}^{t-1}$ and X^t, is performed at the end of grid point calculations, after the computation of the explicit guess has been done using non filtered values X^t for the tendencies at time t. The second part is made at the beginning of the grid point calculations for the following time step, in such a way that the starting point for the leap frog scheme is the completely filtered value (See section 5).

Chapitre 4

Initialisation/Coupling

(08/05/1998)

4.1 Specification of NH variables from an hydrostatic state

The problem of specification of NH variables from an hydrostatic (H) state arises when trying to define initial/coupling files for a NH model with data from an H model output file. In our case the NH model is Aladin, but the H model can be ARPEGE (thus configuration E927) or Aladin (thus configuration EE927). Of course the H model output files contain no information about the NH variables, and some information must therefore be "invented" to be incorporated in the NH files.
Some preliminary works (Bubnova et al., 1995) have shown that the two extra variables introduced by the non-hydrostatism ($\hat{\mathcal{P}}$, $\hat{d}$) adapt themselves to the flow essentially in a local and fast way. The evolution of the flow is not crucially dependent on the value of these fields.
The principle retained here for the specification of the two new variables thus obey to the two following constraints:

- Conservation of the hydrostatic equilibrium initially present in the H state
- Conservation of the acoustic equilibrium initially present in the H state (in the absence of initial diabatic sources of course).

Therefore, all NH simulations basically start with a NH state but in hydrostatic equilibrium, and free of acoustic disturbances. The second constraint specifies that no acoustic energy is introduced in the initial state due to the specification of the two new variables by itself, but of course the acoustic energy potentially introduced by the presence of diabatic sources must be excluded when expressing the constraint. In fact, in the present state of the Aladin NH version this is a false problem since the diabatic sources are eliminated from the $\hat{\mathcal{P}}$ prognostic equation (see future chatper on introduction of diabatism). The two constraints thus have the following mathematical expression:

$$p'(t=0) \equiv 0 \tag{4.1}$$
$$\dot{p'}_{\mathrm{adiab}}(t=0) \equiv 0 \tag{4.2}$$

which write:

$$\hat{\mathcal{P}}(t=0) \equiv 0 \tag{4.3}$$

$$\left(\frac{d\hat{\mathcal{P}}}{dt}\right)_{\text{adiab}}(t=0) \quad \equiv \quad 0 \tag{4.4}$$

The equation (1.47) for the adiabatic evolution of $\hat{\mathcal{P}}$ then writes initially,

$$\left(\frac{d\hat{\mathcal{P}}}{dt}\right)_{\text{adiab}}(t=0) = -\frac{\pi}{\pi^*}\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right)$$

and the second constraint becomes:

$$\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right) \equiv 0$$

replacing with the expression of D_3 given by (1.53), the specification for the initial vertical velocity derivative finally becomes:

$$\left(-g\frac{\partial w}{\partial \eta}\right) = m\frac{RT}{\pi}\left(-\nabla \mathbf{V} - \frac{1}{m}\frac{\pi}{RT}\nabla\phi.\frac{\partial \mathbf{V}}{\partial \eta} - \frac{C_v}{C_p}\frac{\dot{\pi}}{\pi}\right) \tag{4.5}$$

This variable $-g(\partial w/\partial \eta)$ is diagnosed in the `SUBROUTINE POS.F` under the name `ZVDTO`

Chapitre 5

Coding Aspects

(01/08/1997)

5.1 Code Architecture

The non-hydrostatic system is designed, as far as possible, as a correction to the hydrostatic system. This means, in term of code, that the number of specific decks or subroutines should be as small as possible, the non-hydrostatic version being simply viewed as a namelist option. This aspect is more important than apparenlty, since the more hydrostatic code and non-hydrostatic codes are close one to each other, the more accurate and meaningful is the comparison of numerical experiments done with the two separate options. A close comparison of this document and ARPEGE/Aladin notes shows that the degree of similarity between the two systems is effectively very high, resulting in a limited number of specific decks, mainly restricted to low level grid point and spectral subroutines in the main time loop.
The non-hydrostatic dynamics is activated by specifying the namelist parameter:

$$\texttt{LNHDYN} = \texttt{.TRUE.}$$

The Fig. c shows some important stages in the main temporal loop controled by subroutine `STEPO`. The modifications arising from activation of non-hydrostatic version are limitated to two points in the hierarchy of the figure: `CPG` and `ESPC`. Some subroutines called by these two decks are however entirely specific to the non-hydrostatic case, the most important one is `GNHDYN` which treats the non-hydrostatic grid point calculations (instead of `CPDYN` in the hydrostatic case).
Another important part of the modifications take place at the very beginning of the integration, before the temporal loop. This is the set-up phase, in which most of the time-independent features are defined and computed. These computations are made in subroutines the name of which begins by `SU` (for set-up).
Most of explanations depicted in this chapter refer to computations done in the "Vertical Discretisation" and "Time Discretisation" chapters. Only decks modified or new in the non-hydrostatic case are described here.

5.1.1 Set-up calculations

Some set-up decks are devoted to allocation of memory space and initialisation of fields and adresses. Since the number of variables is increased by two in the non-hydrostatic case, these decks have to be modified under the namelist parameter `LNHDYN`.
- `SUELLO`: Allocation of memory space for the variables.
- `SUSLB`: Initialisation of buffers adresses.

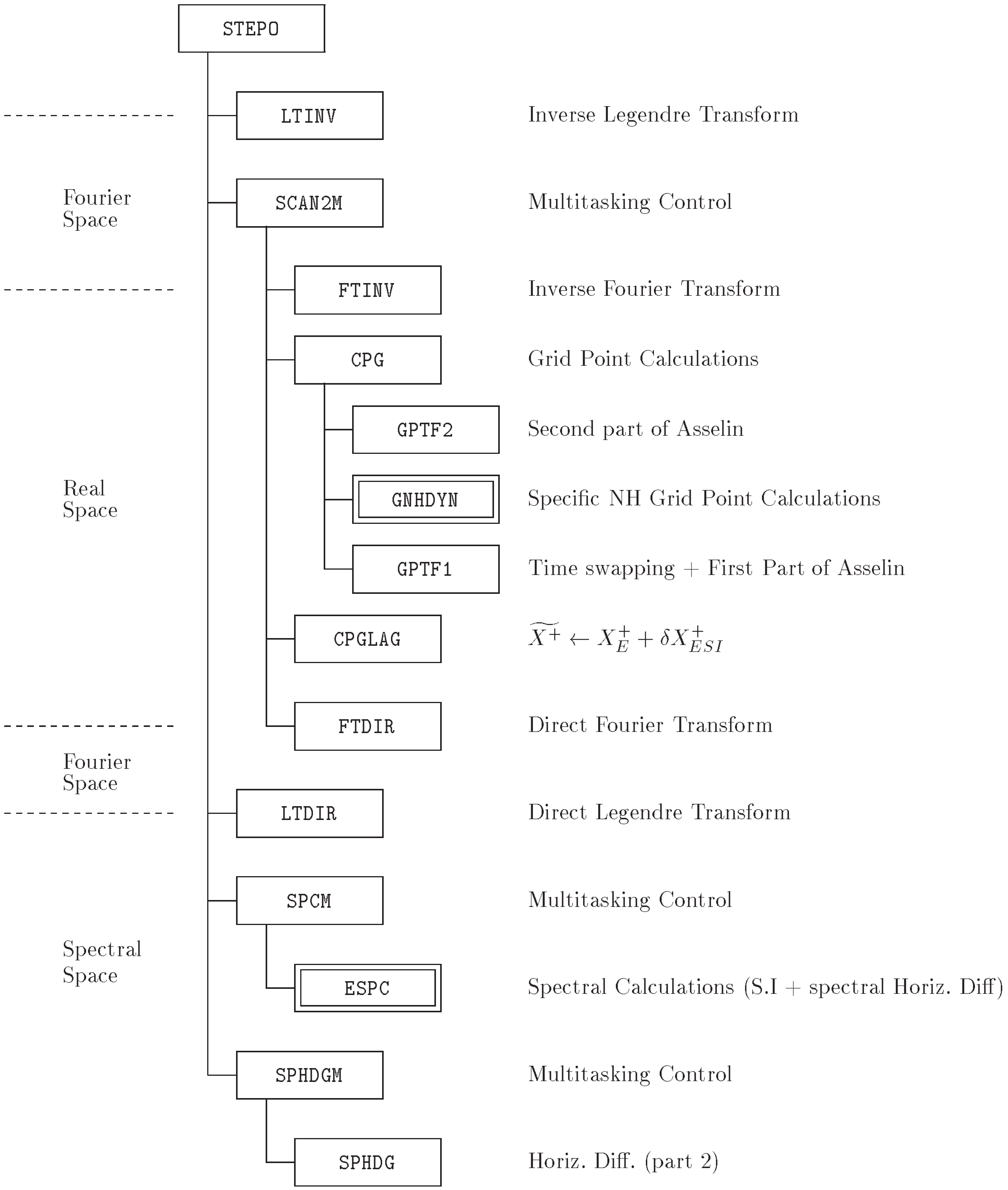


FIG. 5.1: Algorithmic tree of main temporal loop

• SUSC2: Initialisation of multitasking characteristics.
• SUESPEC: Initialisation of spectral fields from ARPEGE/Aladin file.
Some other set-up decks are more specifically concerning dynamical aspects:
• SUEOROG: Initialisation of grid point orography elevation and horizontal derivatives (first and second order). The orography elevation , horizontal gradient and second derivatives are used to respectively compute ϕ through Eq. (1.52), gw_s and ∇w_s through Eq. (2.11).
• SUEHEL: Initialisation of inverse semi-implicit operator $\mathbf{B}^{-1}$ for each spectral component, in Eq. (3.10).
• SUEDYN: Initialisation of namelist control parameter LNHDYN for non-hydrostatic dynamics.
• SUDYN: Control for initialisation of dynamic parameters. Initialisation of basic state vertical characteristics through subroutines GPPREH and GPXYB.

5.1.2 Grid-point calculations

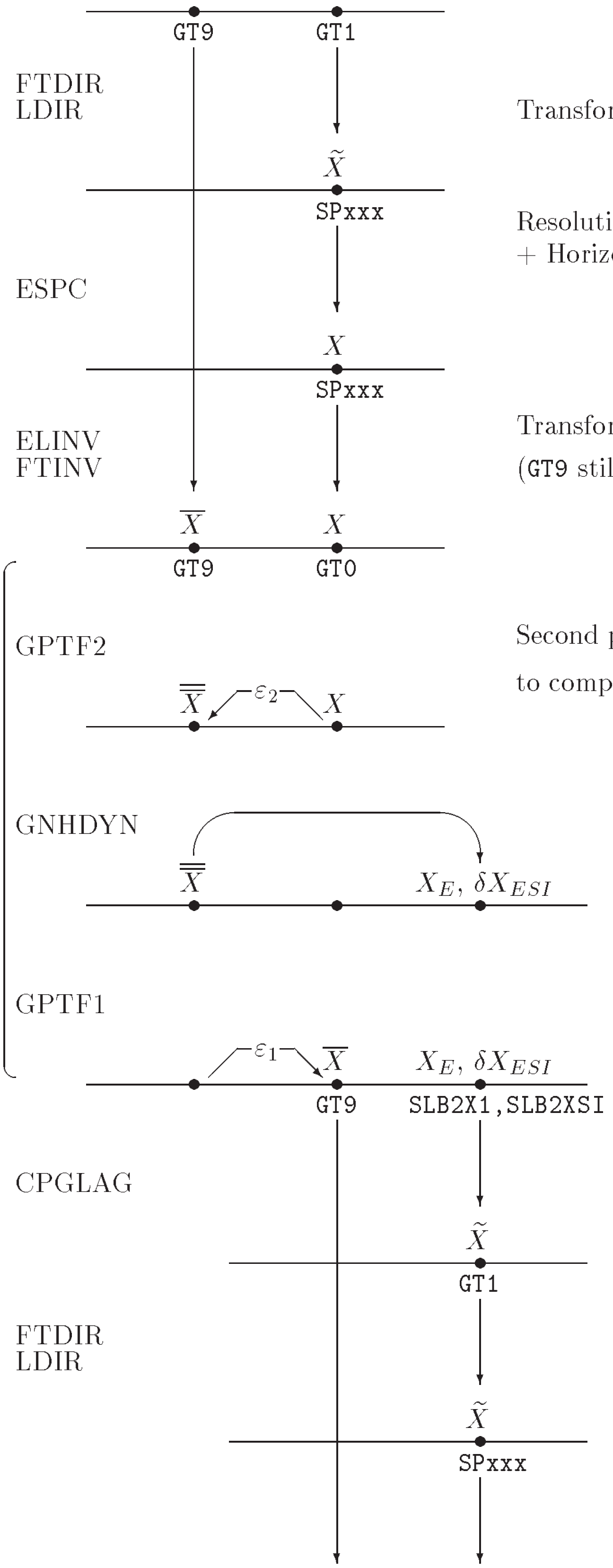
The grid point calculations mainly consist in evaluation of the explicit guess and the explicit part of semi-implicit correction which are used as right hand side terms in spectral computations. The Asselin time filter is also performed in grid point, but splitted into two parts. The main steps of grid point calculations are as follow (in chronological order):
a) Subroutine CPG:
• "Second" part of time filter in subroutine GPTF2: $X^-_{\text{filtered}} \leftarrow X^-_{\text{partially filtered}} + \varepsilon_2 X^0$.
• Multiplication by the map factor for vectorial fields (i.e. fields associated to wind and/or to a ∇ operator) in subroutine GPMPFC.
• Computation of hydrostatic pressure at half levels (interfaces) in the subroutine GPPREH.
• Computation of discretisation parameters α_l, $\delta\pi_l$,... in the subroutine GPXYB.
• Computation of hydrostatic pressure at full levels (layers) in the subroutine GPPREF.
• Computation of thermodynamics parameters R, R', C_p,... in the subroutine GPRCP.
• Computation of variables associated to the continuity equation integral $m\dot{\eta}$, ω/π,... in the subroutine GPCTY.
• Computation of ϕ_l and $\phi_{\tilde{l}}$, in the subroutine GPGEO.
• Computation of the pressure gradient term in the subroutine GNHGRP.
• Call physical package in the subroutine APLPAR.
• Call dynamical package in the subroutine GNHDYN. Computation of X^+_E and δX^+_{ESI}.
• "First" part of time filter in subroutine GPTF1 $X^0_{\text{partiallyfiltered}} \leftarrow (1-\varepsilon_1-\varepsilon_2)X^0+\varepsilon_1 X^-$. and swapping $X^- \leftarrow X^0$ in the subroutine GPTF1. The net result of these two operations is: $X^-_{\text{partiallyfiltered}} \leftarrow (1-\varepsilon_1-\varepsilon_2)X^0+\varepsilon_1 X^-$.
b) Subroutine CPGLAG:
• Summation of all RHS part of the SI scheme $\widetilde{X^+} \leftarrow X^+_E + \delta X^+_{ESI}$. in the subroutine GNHDYN.
• Storage in GT1 in subroutine GPENDTR

5.1.3 Spectral calculations

The spectral computations occur in the subroutine ESPC. They essentially concern the solution of semi-implicit and horizontal diffusion computations. This occurs essentially the same way as in the hydrostatic case (at least for the code architecture point of view). The difference with hydrostatic case only come from the nature of operator to build and invert.

5.2 Data Flow

The implementation of the data flow in the discretised time step appears in Fig.(5.2).



Beginning the time step with global grid point array `GT1` containing RHS of Helmholtz SI equation: $\widetilde{X}^t$

Transformation to spectral array `SPxxx`

Resolution of SI provides X^t in `SPxxx`
+ Horizontal Diffusion

Transformation to grid point array X^t `GT0`
(`GT9` still contains partially time-filtered $\overline{X}^{t-\Delta t}$)

Second part of time filter: update of `GT9` to completely filtered value $\overline{\overline{X}}^{t-\Delta t}$

Computation of $X_E^{t+\Delta t}$ in `GT1` through leap-frog.
Computation of $\delta X_{ESI}^{t+\Delta t}$.

First part of time filter: *replace* `GT9` by partially filtered value $\overline{X}^{t-\Delta t}$.

Computation of RHS of Helmholtz SI equation: $\widetilde{X}^{t+\Delta t} = X_E^{t+\Delta t} + \delta X_{ESI}^{t+\Delta t}$.
Storage in `GT1`.

FIG. 5.2: Implementation of discretised time step (left brace indicates CPG calculations)

5.2.1 Data flow for iteration of SI scheme

The whole iteration process is controled at the level of CNT4. At this place are made the calls to the time-step temporal loop STEPO, with the CDCONF parameter controlling what is to be done in this loop. If no iteration is specified (NSITER=0), STEPO is called only once. In the case of the iterated SI scheme (NSITER > 0), STEPO is called NSITER times in a loop which really is the kernel of the iterative process, and which is described by the index JSITER = (0,...,NSITER). The values of the control parameter CDCONF is given in the following table (star denote non relevant characters):

```
NSITER = 0                              ***A****A

NSITER > 0   JSITER = 0                 ***A****N
             0 < JSITER < NSITER        ***N****Y
             JSITER = NSITER            ***N****A
```

Of course, in the case NSITER=1, which is used practically, only second and fourth lines are used.
it should be noted that the unique STEPO call in the case NSITER=0 is not identical to the first one when NSITER>0

The CDCONF values basically inhibit or activate some parts of code inside STEPO:

Subroutine	value of CDCONF(4) for activation	value of CDCONF(9) for activation
SC2RDG(ZGT0)	≠N	
SC2RDG(ZGPNH)	=N	
CPG	≠N	
ESTRCP	=A and NSITER>0	
SCRWRG(ZGPNH)	=A and NSITER>0	
CPGLAG	≠N	
GNHSIMLAG	=N	
GNHSIM	=N and NSITER>0	
SC2WRG(ZGT1)	≠N	
ESPAWR		=N
ESPAY	=N	=A or =Y

The algorithmic process of the iteration is presented on Figs.(5.3) and (5.4) for JSITER=0 and JSITER=1 respectively, for the cycle AL06.
Here follows a short depiction of the data flow specific to the iterative process:

Starting in spectral space for JSITER=0, the two (x and y) inverse Fourier transforms are done normally, ending with the array PGT0() in grid-point space, containing X^0.

Then CPG is executed giving $X_E^{+(0)}$ in PB2(...X...) and $X_{ESI}^{+(0)}$ in PB2(...XSI...). At the very end of CPG, the "strangers" are "strangely" stored in the arrays normally devoted to other fields (namely: $\hat{\mathcal{P}}^0$, $\hat{d}^0$, $\nabla\hat{\mathcal{P}}^0$ and $\partial\hat{d}^0/\partial x$) which fortunately are supposed not to be used afterward.

As soon as exiting CPG, these strangers are transferred to a safer place (i.e. in a specific array ZGPNH()) in ESTRCP. Then, in a call to SC2WRG they are stored into the buffer GPNHBUF().

The remaining of grid-point computations performs normally, as well as the spectral transform, which gives the arrays `SP*()` (or the big general array `SPA3()`) containing $\widetilde{X^{+(0)}}$ data.

Before doing the SI inversion, the whole $\widetilde{X^{+(0)}}$ generic array `SPA3()` is copied in the array `SPA3BC()` kept "in the fridge" for later use in next iteration (the suffix "BC" probably stands for "Before Computation").

Then the SI inversion is done normally in `ESPC` and the iteration `JSITER=0` is finished

Starting in spectral space for `JSITER=1`, the two (x and y) inverse Fourier transforms are one more time done normally, ending with the array `PGT0()` in grid-point space. But now, it contains $X^{+(0)}$.

A call to `SC2RDG` then reads the buffer `GPNHBUF()` into the array `ZGPNH()`, and all normal grid-point computations are skipped.

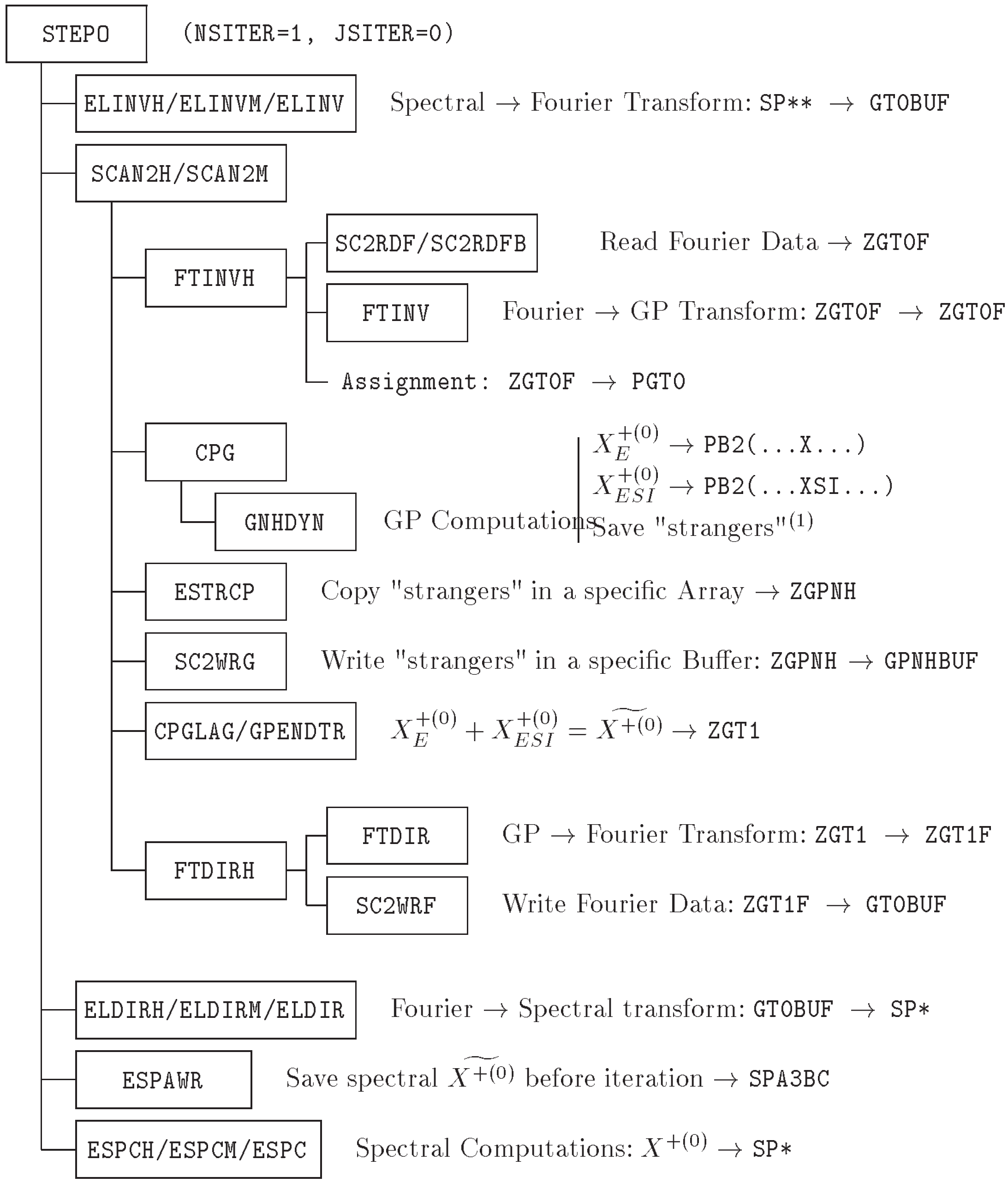
In `GNHSIM` the time-averaged Y term $2\beta\Delta t.(Y^{+(0)} + Y^{-} - 2Y^{0})$ is computed and stored by overwiting the $\hat{\mathcal{P}}^{+(0)}$ field (i.e. in `PB2(...MSLB2PD1...)` buffer).

In `GNHSIMLAG`, the latter field is transferred to the array `ZGT1(...PD1...)` containing the fields to be spectrally transformed.

The spectral transforms are then executed, ending with the arrays `SP*()` (or the big general array `SPA3()`) containing $\widetilde{X^{+(0)}}$ data (don't forget that the $\hat{\mathcal{P}}^{+(0)}$ array still contains in fact the field $2\beta\Delta t.(Y^{+(0)} + Y^{-} - 2Y^{0})$).

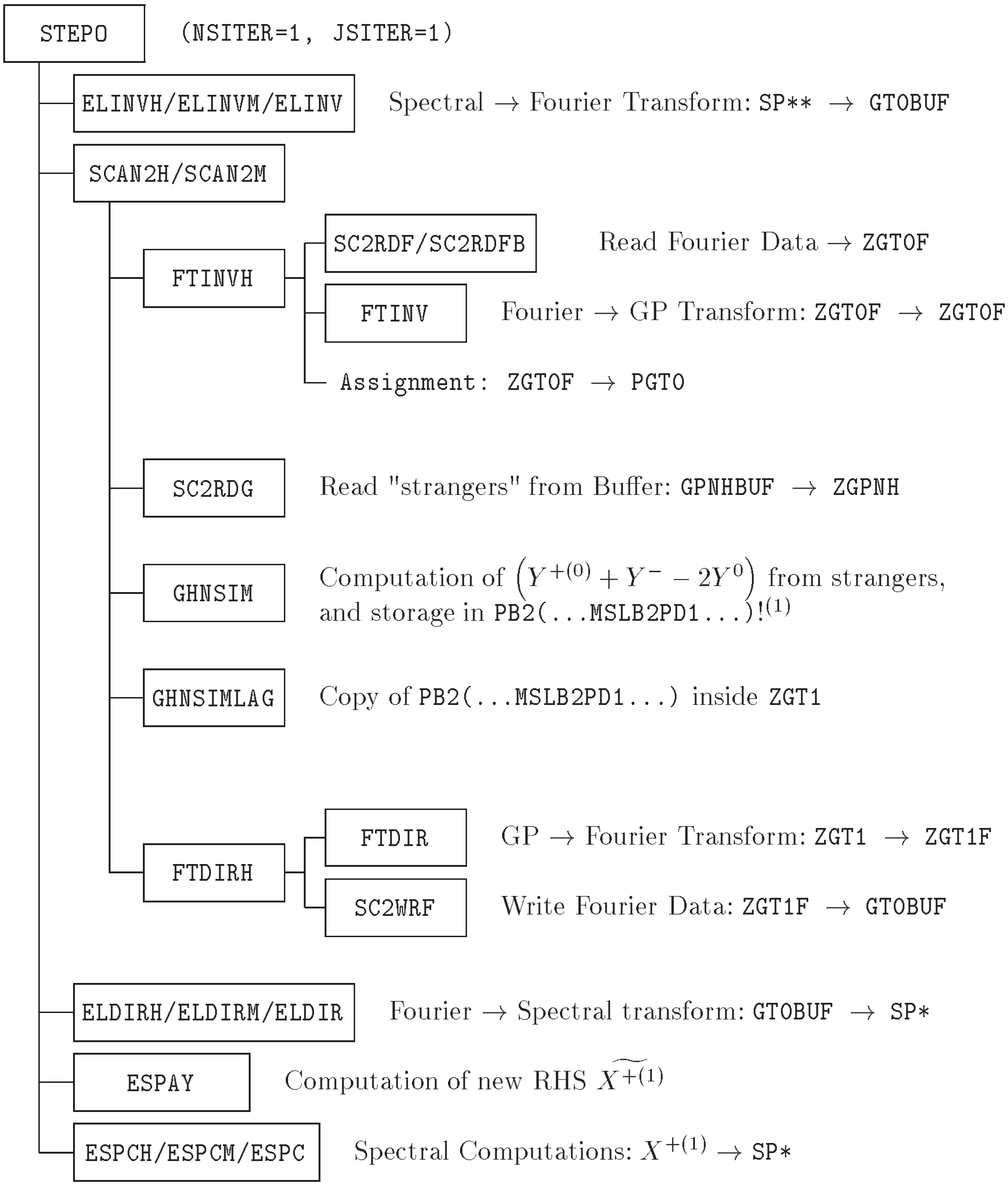
In `ESPAY`, the field $2\beta\Delta t.(Y^{+(0)} + Y^{-} - 2Y^{0})$ is copies in the array `ZYITER()`, and the old `SPA3BC()` is used to completely overwrite `SPA3()`, which now contains the non-iterated original RHS of the SI scheme $\widetilde{X^{+(0)}}$. Then the iterated RHS $\widetilde{X^{+(1)}}$ are computed by combining $\widetilde{X^{+(0)}}$ and $2\beta\Delta t.(Y^{+(0)} + Y^{-} - 2Y^{0})$ as in Eq. (3.17) – (3.20).

Then the SI inversion is done normally in `ESPC` and the iteration `JSITER=1` is finished with $X^{+(1)}$ in `SP*()`



(1) "Strangers" are the 5 fields which must be transmitted between iterations:
Namely: $2Y^0 - Y^-$, R'^0, $\nabla\phi^0$ (2 fields), and discretisation weight.
They are (very) temporarily stored in the $\hat{\mathcal{P}}^0$, $\hat{d}^0$, $\nabla\hat{\mathcal{P}}^0$ and $\partial\hat{d}^0/\partial x$ arrays.

FIG. 5.3: Algorithmic tree of first (JSITER=0) iteration of the SI scheme



(1) The storage of this field is therefore done by temporarily overwriting on $\hat{\mathcal{P}}^{+(0)}$! But all the original $X^{+(0)}$ fields will be retrieved in the saved array SPA3BC which has been performed during the previous iteration (in ESPAWR).

FIG. 5.4: Algorithmic tree of last (JSITER=1) iteration of the SI scheme

5.3 Name Correspondance for Global Variables

$$
\begin{array}{rcl}
\texttt{SEVE(subr.)} & \longrightarrow & \mathbf{L}^* \\
\texttt{SIALPH(JLEV)} & \longrightarrow & \alpha_l^* \\
\texttt{SIDELP(JLEV)} & \longrightarrow & \delta\pi_l^* \\
\texttt{SIDPHI(JLEV)} & \longrightarrow & RT^*\delta_l^* = -\delta\phi^*! \\
\texttt{SILNPR(JLEV)} & \longrightarrow & \delta_l^* \\
\texttt{SIPR} & \longrightarrow & \pi_s^* \\
\texttt{SIRDEL(JLEV)} & \longrightarrow & (1/\delta\pi_l^*) \\
\texttt{SITLAF(JLEV)} & \longrightarrow & \pi_l^* \\
\texttt{SITLAH(JLEV)} & \longrightarrow & \pi_{\tilde{l}}^* \\
\texttt{SITR} & \longrightarrow & T^* \\
\texttt{ZA(JLEV)} & \longrightarrow & \mathsf{A}_l^* \\
\texttt{ZB(JLEV)} & \longrightarrow & \mathsf{B}_l^* \\
\texttt{ZC(JLEV)} & \longrightarrow & \mathsf{C}_l^* \\
 & \longrightarrow & A_{\tilde{l}} \\
 & \longrightarrow & B_{\tilde{l}} \\
 & \longrightarrow & C_{\tilde{l}}
\end{array}
$$

Chapitre 6

References

Tanguay, M., A. Robert and R. Laprise, 1990: A semi-implicit semi-lagrangian fully compressible regional forecast model. *Mon. Wea. Rev.*, **118**, 1970–1980.
Ikawa
Cote and staniforth

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