Isoperimetric problem (2)

Lagrange multipliers method can be applied :

$$d\int_0^{2\pi}\left[\frac{1}{2}f(\vartheta)^2-\lambda\sqrt{f'(\vartheta)^2+f(\vartheta)^2}\right]d\vartheta=0$$

Then the Euler-Lagrange equation is applied :

$$\frac{\partial F}{\partial f'}=-\lambda\frac{f'}{\sqrt{f'^2+f^2}} \qquad \frac{\partial F}{\partial f}=f-\lambda\frac{f}{\sqrt{f'^2+f^2}}$$

Isoperimetric problem (3)

$$\frac{d}{d\vartheta}\frac{\partial F}{\partial f'} = \frac{\lambda f}{(f'^2+f^2)^{\frac{3}{2}}}(f'^2 - f' f'')$$

Then : $$\frac{\lambda f}{(f'^2+f^2)^{\frac{3}{2}}}(f'^2 - f' f'') = f - \lambda \frac{f}{\sqrt{f'^2+f^2}}$$

$$\lambda = \frac{(f'^2+f^2)^{\frac{3}{2}}}{2f'^2+f^2-f'f''}$$

This is the expression of the curvature in polar coordinates.

The solution has a constant curvature : it is a circle

Lagrange's equations

Hamilton's principle of least action :

The system define by the Newton's equations of dynamics

$$\frac{d}{dt}(m_i \dot{r}_i) + \frac{\partial U}{\partial r_i} = 0$$

Coincide with extremals of the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} L dt$$

Where L = T-U is the difference between
the kinetic and potential energy

Since U = U(r) and $T = \sum m_i \frac{r_i^2}{2}$ we have $\frac{\partial L}{\partial \dot{r}_i} = m_i \dot{r}_i$ and $\frac{\partial L}{\partial r_i} = -\frac{\partial U}{r_i}$

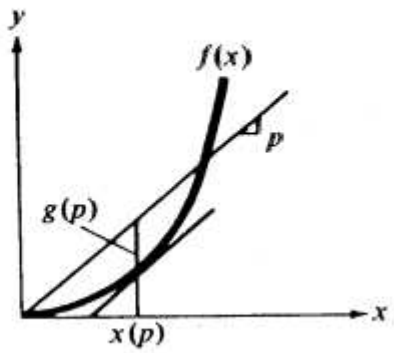
Legendre transformation

Let y=f(x) be a convex function, f''(x) >0

The Legendre transformation of the function f is a new function g Of a new variable p, which is constructed in the following way.

Let p be a given number. Consider the straight lin y=px. We take the point x=x(p) at which the curve is farthest from the straight line in the vertical direction : for each p the function px – f(x) = F(p,x(p)) has a maximum with respect to x at the point x(p).

Now we define g(p) = F(p,x(p)). The point x(p) is defined by the Extremal condition $\partial F/\partial x = 0$, i.e., f'(x) = p.



Legendre transformation (2)

Exemples :

Let $f(x) = x^2$. Then $F(p,x) = px - x^2$, $x(p) = \frac{1}{2}\, p$, $g(p) = \frac{1}{4}\, p^2$

Let $f(x) = mx^2/2$. Then $g(p) = p^2/2m$

Let $f(x) = x^\alpha/\alpha$. Then $g(p) = p^\beta/\beta$, where $(1/\alpha) + (1/\beta) = 1$

Involutivity

The Legendre transformation is involutive, i.e., its square is the identity: if under the Legendre transformation f is taken to g, then the Legendre transformation of g will again be f

Hamilton's equations (1)

Let q_i generalized coordinates.

We consider the system of Lagrange's equations : $\dot{p} = \frac{\partial L}{\partial q}$

Where : $p = \frac{\partial L}{\partial \dot{q}}$

The Legendre transform of $L(q, \dot{q}, t)$ with respect to $\dot{q}$ is the function $H(p) = p\,\dot{q} - L(\dot{q})$

This function is called the hamiltonian

The total differential of the hamiltonian is

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt$$

Hamilton's equation (2)

The total differential of the hamiltonian is also :

$$dH = \dot{q}\, dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt$$

Both expressions for dH must be the same. Therefore,

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p}$$

We obtain Hamilton's equations

Liouville's theorem

The 2n-dimensional space with coordinate $p_1, \ldots, p_n ; q_1, \ldots, q_n$ is called phase space.

The phase flow is the one parameter group of transformation of phase space : $g^t : (p(0),q(0)) \rightarrow (p(t),q(t))$
Where p(t) and q(t) are solutions of Hamilton's system of equations

The phase flow preserves volume, for any region D we have :

Volume of $g^t D$ = volume of D

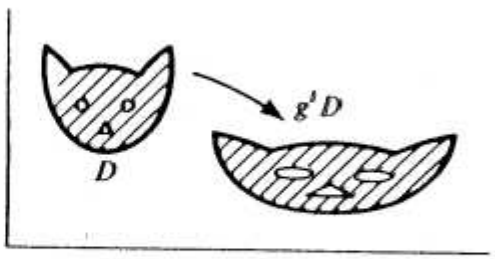


Figure 48 Conservation of volume

Liouville's theorem (3)

In the meteorological case, any model can be written as a set of N ordinary differential equations :

$$\dot{X} = \Phi\left[X\ (t)\right]$$

The theorem of Liouville can be introduced as the continuity Equation for the pdf of the state vector X(t)

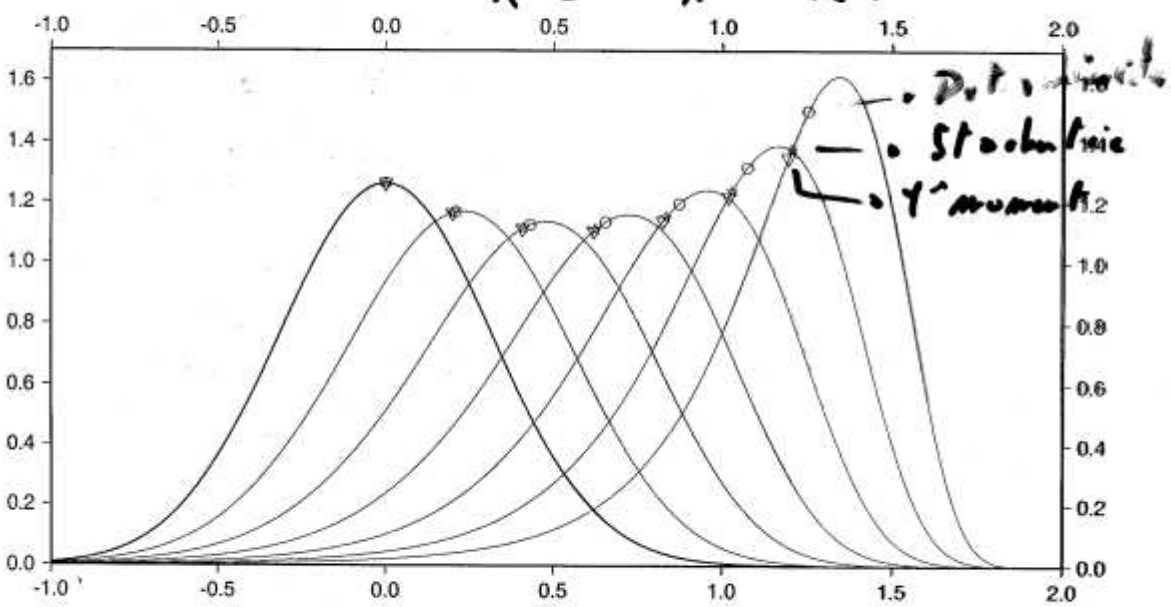
$$\frac{\partial \rho(X,t)}{\partial t} + \sum_{k=1}^{N} \frac{\partial}{\partial X_k}\left[\rho(X,t)\dot{X}_k(X,t)\right] = 0$$

Liouville's theorem (2)

- The state of a system is fully described by a point in phase space and the temporal evolution
- These points trace out a trajectory in phase space
- In analogy with statistical mechanics a very large number of realizations may satisfies the continuum hypothesis in phase space.
- The theorem of Liouville can be formulated as a continuity equation for the Probability Density Function

Evolution of an initial Gaussian PDF in the case of a Riccati equation

$$\dot{X} = a X^2 + bX + c$$



Improvement of the spectral representation of the Earth topography with a variational method

- Orography representation in spectral models has always led to problems

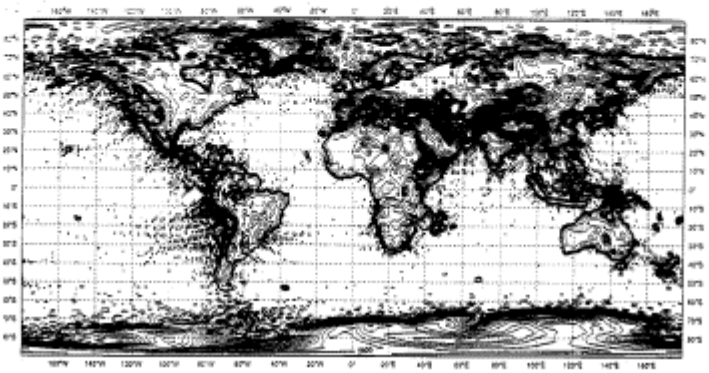
- To avoid aliasing errors in the nonlinear computations, the gridpoint space contains more degrees of freedom than the spectral space

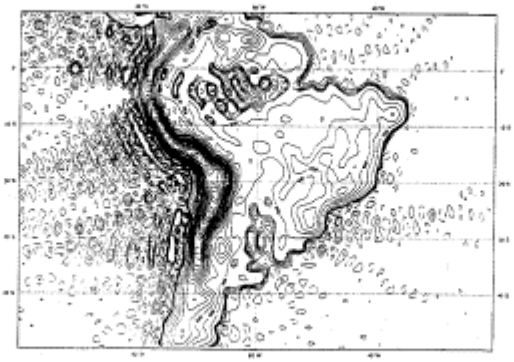
- This leads to the problem of orography approximation with Gibss waves in the vicinity of areas affected by strong variation of altitude

T 126

$]-\infty, 90]$ ⇒ 15m

$[200, +\infty[$ ⇒ 200m





Spectral representation

In ARPEGE : $$f(\lambda,\mu)=\sum_{n=0}^{N}\sum_{m=-n}^{n} f_n^m Y_n^m(\lambda,\mu)$$

Where, $$Y_n^m(\lambda,\mu)=P_n^m(\mu)e^{im\lambda}$$

In ALADIN : $$f(x,y)=\sum_{n=-N}^{n=N}\sum_{m=-M(n)}^{n=M(n)} f_n^m e^{inx} e^{imy}$$

Classical orthogonal projection

The classical orthogonal projection minimize the very simple following cost functions :

In ARPEGE : $$F = \sum_{\lambda} \sum_{\mu} \omega(\mu) \left[\tilde{R}(\lambda,\mu) - R(\lambda,\mu) \right]^2$$

In ALADIN : $$F = \sum_{x} \sum_{y} \left[\tilde{R}(x,y) - R(x,y) \right]^2$$

In these formulas, R is the real orography and $\tilde{R}$ the spectrally fitted orography to determine

How to reduce the small scale noise ?

- It is not possible to eliminate the representation problem links to the degrees of freedom of the grid point space

- However, a 100-m error can be considered negligible over high mountains but disastrous over sea.

- The aim is to move the errors over the mountainous area and, in ARPEGE, far away from the pole of maximum resolution

First cost function

The first cost function has been proposed for ARPEGE and applied with success in ALADIN

$$F=\sum_{\lambda}\sum_{\mu}\omega(\mu)\left[\frac{|\tilde{R}(\lambda,\mu)-R(\lambda,\mu)|}{HDIM}\right]^{W(\lambda,\mu)}$$

Where :

$$W(\lambda,\mu)=W_{min}+(W_{max}-W_{min})\left[1+\frac{c^2-1}{c^2+1}\left(\frac{\mu-1}{2}\right)\right]\exp\left[\frac{-|R(\lambda,\mu)|}{HMIN}\right]$$

First cost function

The components of the gradient of such a cost function are :

$$(\nabla F)_{\lambda_i,\mu_i} = W(\lambda_i,\mu_i)\,\omega(\mu_i)\,\frac{\tilde{R}(\lambda_i,\mu_i) - R(\lambda_i,\mu_i)}{HDIM^2}\left[\frac{|\tilde{R}(\lambda_i,\mu_i) - R(\lambda_i,\mu_i)|}{HDIM}\right]^{W(\lambda_i,\mu_i)-2}$$

The gradient in spectral space is then computed by applying the adjoint of the inverse spectral transform

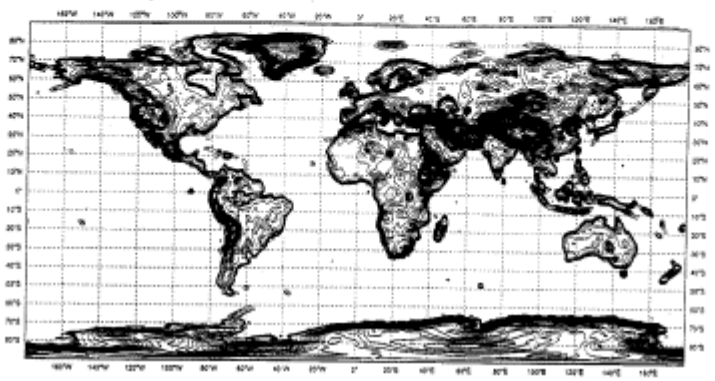
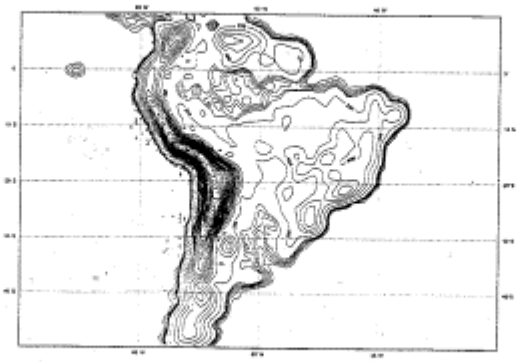
$$(\nabla F)_n^m = \frac{1}{4\pi}\sum_{k=1}^{K}\int_O^{2\pi}(\nabla F)_{\lambda_k\mu_k}\,P_n^m(\mu_k)\,e^{im\lambda}\,d\lambda$$

Second cost function

A second cost function has been proposed by Jerczynski and works well in ALADIN when the domain does not contain sea area

$$F = \sum_x \sum_y \left[W(x,y)(\tilde{R}(x,y) - R(x,y))^2 + (Qconst\,|\tilde{R}(x,y) - R(x,y)|)^{Qpower} \right]$$

T 126 After Anti-Gibbs

$]-\infty, 90] \Rightarrow 15\,m$

$[200, +\infty] \Rightarrow 200\,m$