Schedule of the lectures

- First : a « cutural » lecture with historical digression about variational calculus.
- Then an introduction to the optimal estimation theory,
- and to variational data assimilation in meteorology.
- Finally a lecture on the minimization algorithms.

History of the calculus of variations

Ancient Egyptians knew (empirically) that :

- The shortest path between two points is a straigt line
- The circle is the geometrical figure with the largest enclosed area inside a given perimeter

History of the calculus of variations

Ancient greece

- Birth of modern mathematics. There mathematicians did not only ask how, but also why.
- Archimedes proved that the shortest path joining two points is a straight line.
- Zenodoros proved that the area of a polygon is always smaller than the area of a circle with the same perimeter.

History of the calculus of variations

Birth of modern variational calculus: The brachystochrone problem (Johann Bernoulli 1696)

- A point mass glides without friction along a curve joining a point A to a lower point B: Which curve gives the shortest time of travel ?

- The solution is a simple transcendental curve: the cycloid

History of the calculus of variation

- The various methods used in solving the brachystochrone problem were adhoc.
- The next step is the development of variational calculus as a general theory.
- The names Lagrange (1736-1813) and Euler (1707-1783) are inextricably linked with this step.

History of the calculus of variations

- The coming together of ideas and methods from algebra, geometry, topology and analysis produced a new branch of mathematics: functional analysis.

- Variational calculus must be regarded as the oldest part of functional analysis.

Variational principles

- Generalities
- Euler-Lagrange equation
- Classical variational problems
- Lagrange's equations
- Legendre transformations
- Hamilton's equations
- Liouville's theorem

Generalities (1)

The calculus of variations is concerned with the extremals of functions whose domain is an infinite-dimensional space : the Space of curves. Such functions are called functionals.

An exemple of a functional is the length of a curve in the euclidian plane :

If $$\gamma = \left\{ (t,x) : x(t) = x, t_0 \le t \le t_1 \right\}$$

then : $$\Phi(\gamma) = \int_{t_0}^{t_1} \sqrt{(dx)^2 + (dt)^2} = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2}\, dt$$

Generalities (2)

We consider an « approximation » $\gamma' = \gamma + h$

Then a functionnal Φ is called differentiable if :

$$\Phi(\gamma + h) = F + R$$

Where F depends linearly on h and

$$R(h, \gamma) = O(h^2)$$

F(h) is called the differential

Euler-Lagrange equation (1)

Expression of the differential F :

We define a functional by $\Phi(\gamma)=\int_{t_0}^{t_1} L(x(t),\dot{x}(t),t)\,dt$

Then, $\Phi(\gamma+h)-\Phi(\gamma)=\int_{t_0}^{t_1}\left[L(x+h,\dot{x}+\dot{h},t)-L(x,\dot{x},t)\right]dt$

$$\Phi(\gamma+h)-\Phi(\gamma)=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right]dt+O(h^2)$$

$$F(h)=\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right)dt$$

Euler-Lagrange equation (2)

Integrating by parts, we find that :

$$F(h) = \int_{t_0}^{t_1} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \right] h \, dt + \left[\left(\frac{\partial L}{\partial \dot{x}} h \right) \right]_{t_0}^{t_1}$$

The curve γ is an extremal of the functional ϕ if and only if :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

This equation is called Euler-Lagrange equation

Constrained minimisation problems (Method of Lagrange multipliers)

Find the extremal of the function $f(x_1,\ldots,x_n)$ under the constraint $g(x_1,\ldots,x_n) = 0$ is equivalent to find the extremal of $f - \lambda g$ where λ is a constant

The Euler-Lagrange equation can be used with the new function

Classical variational problems

Example of application, we search the extremals of length

We have : $L = \sqrt{1 + \dot{x}^2}$ ➔ $\frac{\partial L}{\partial x} = 0$ and $\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}$

Then, $\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} = c$ $\dot{x} = c_1$

$$x = c_1 t + c_2$$

The extremals of length are straight lines

An important remark

The condition for a curve γ to be an extremal of a functional does not depend on the choice of coordinate system.

For example, the same functional – length of a curve – is given in cartesian and polar coordinates by the different formulas

$$\Phi_{cart} = \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt \qquad \Phi_{pol} = \int_{t_0}^{t_1} \sqrt{\dot{r}^2 + r^2 \dot{\phi}^2}\, dt$$

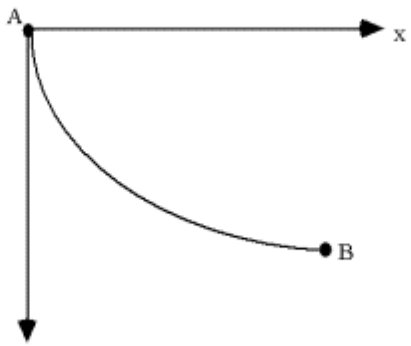
The extremals are the same – straight lines in the plane. The equations of the lines in cartesian and polar coordinates are given by different functions

However, both these satisfy the Euler-Lagrange equation

Classical Variational Problems

- The Brachystochrone:

  We have to determine the curve q(x) passing through two points, such that a point of mass m=1, travelling from A to B under the inluence of gravity, reaches B in the shortest possible time.



The Brachystochrone problem

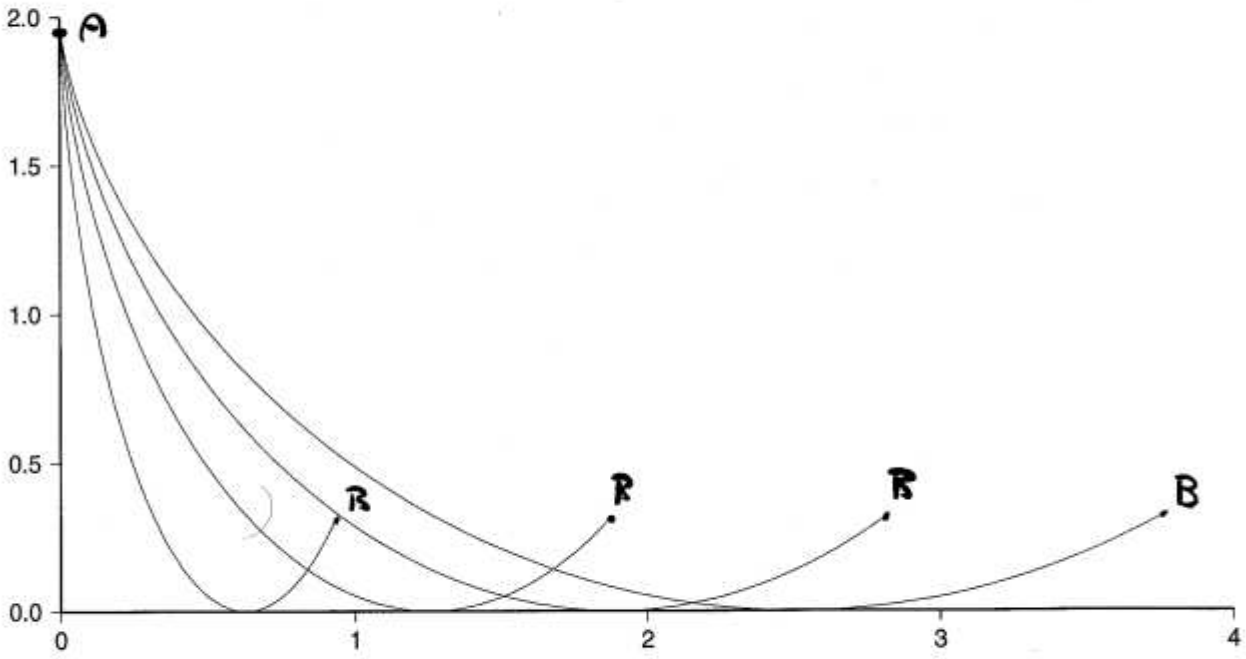
The magnitude of the velocity at an arbitrary point (x,q(x)) is :

$$v=\frac{ds}{dt}=\sqrt{2gx(t)}$$

The point mass needs a time :

$$T(q)=\int_0^b \frac{ds}{v}=\int_0^b \sqrt{\frac{1+q'^2(x)}{2gx}}\,dx$$

Brachystochrone problem



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Geodesics problem

The problem consists in determining the shortest curve between two given points A and B on a given surface Σ

The case is obviously a variational problem subject to a constraint

Isoperimetric problem

Find a closed curve with a given perimeter
which inclose the larger area

In polar coordinate r = f(θ) ➔ $dS = \frac{1}{2} r^2 \, d\vartheta$

$$S = \int_0^{2\pi} \frac{1}{2} f^2(\vartheta) \, d\vartheta$$

The problem is to maximise S under the constraint :

$$L = \int dl = \int_0^{2\pi} \sqrt{(f'(\vartheta))^2 + (f(\vartheta))^2} \, d\vartheta$$