D. CHOICE OF A DIGITAL FILTER

D.1. Available non-recursive digital filters in ALADIN

Definition of non-recursive filters

➤ Discrete formulation of the filtered state

$$\overline{x}_n = \sum_{k=-N}^{+N} h_k \, x_{n-k} \quad , \quad h_k = h_{-k} \quad , \quad x_k = x\left(k\,\Delta t\right)$$

➤ Discrete formulation of the response function

$$H(\omega) = h_0 + 2\sum_{k=0}^{N} h_k \cos(k\omega\Delta t) \quad , \quad \omega = \frac{2\pi}{T}$$

H is real

➤ Notations

(h_k) : weights

Δt : time-step

$2N+1$: order of the filter

$\mathcal{T}=2N\Delta t$: time-span (integration length)

T : period

The "Ideal low-pass" filter

➤ Weights, for a chosen cut-off period T_c :

$$h_n = \frac{\sin(n\omega_c\Delta t)}{n\pi} \qquad , \quad T_c = 2\pi/\omega_c \qquad , \quad -N \le n \le N$$

➤ Ideal response ($N = \infty$) :

$\mathbf{H(\omega) = 1}$ if $0 \le \omega \le \omega_c$ $\quad (T \ge T_c)$

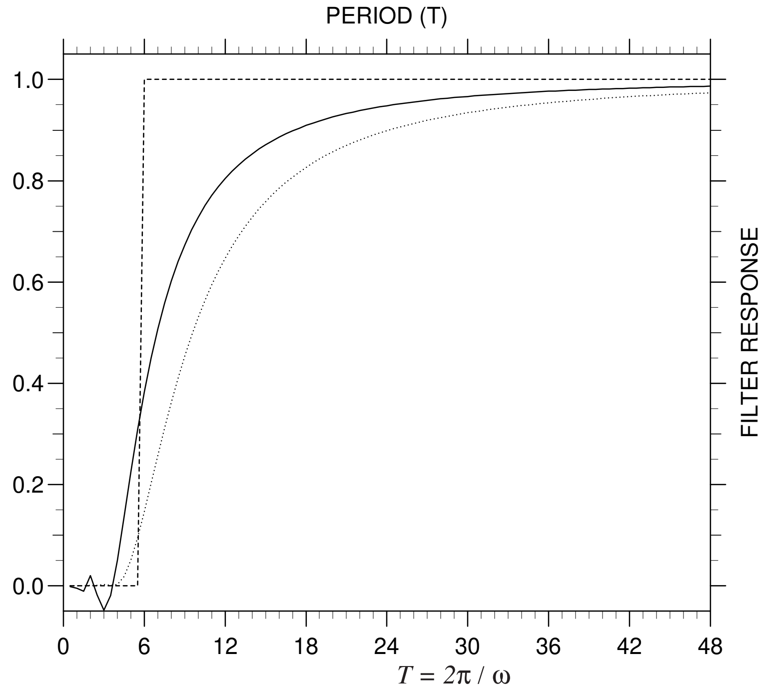
$\mathbf{H(\omega) = 0}$ if $\omega_c < \omega \le \pi / \Delta t$ $\quad (2\Delta t \le T < T_c)$

➤ Minimum time-span : $\mathcal{T}_{\min} = \sqrt{3} / \pi \cdot T_c$

➤ Effective response :

unpleasant oscillations in the stop-band and some damping in the pass-band ...

➤ Example :



LEGEND :

Full line : effective response

Dashed line : ideal response

Dotted line : squared response

FILTER CHARACTERISTICS :

Cut-off period : $T_c = 6\,\mathrm{h}$
Time-span : $\mathcal{T} = 6\,\mathrm{h}$
Time-step : $\Delta t = 450\,\mathrm{s}$
Order : $2N+1 = \mathcal{T} / \Delta t + 1 = 49$

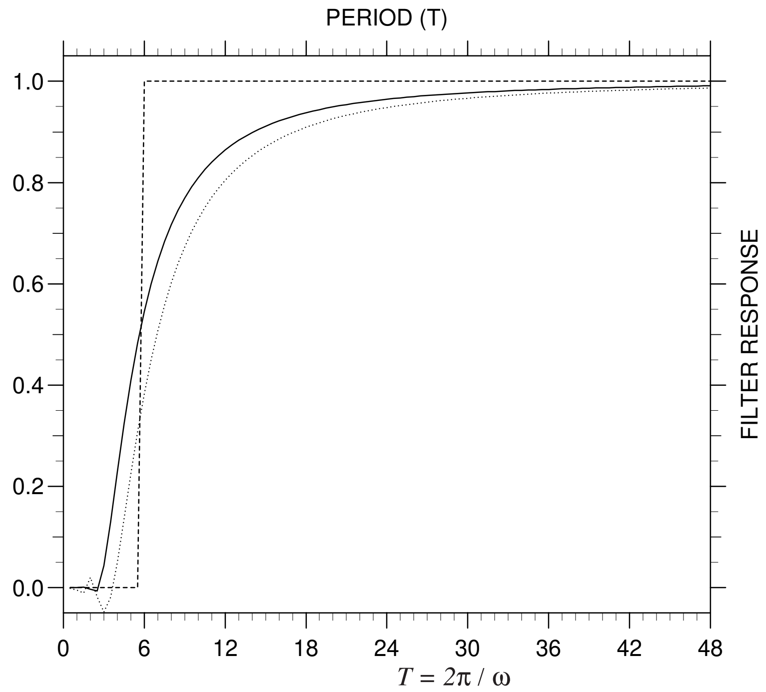
The "Ideal low-pass" filter with a "Lanczos" window

➤ Weights, for a chosen cut-off period T_c :

$$h_n = \frac{\sin(n\omega_c \Delta t)}{n\pi} \times w_n \,, \qquad w_n = \frac{\sin(n\pi/(N+1))}{n\pi/(N+1)} \qquad , \quad T_c = 2\pi/\omega_c \qquad , \quad -N \le n \le N$$

The weights of the "Ideal low-pass" filter (h_n) are modulated by a "window" (w_n), in order to damp oscillations in the stop-band (i.e. below T_c) and reduce attenuation in the pass-band.

➤ Example :



LEGEND :

Full line : effective response

Dashed line: theorical response

Dotted line : raw response-no window

FILTER CHARACTERISTICS :

Cut-off period : $T_c = 6\,\mathrm{h}$
Time-span : $\mathcal{T} = 6\,\mathrm{h}$
Time-step : $\Delta t = 450\,\mathrm{s}$
Order : $2N+1 = \mathcal{T} / \Delta t + 1 = 49$

The "Optimal" filter

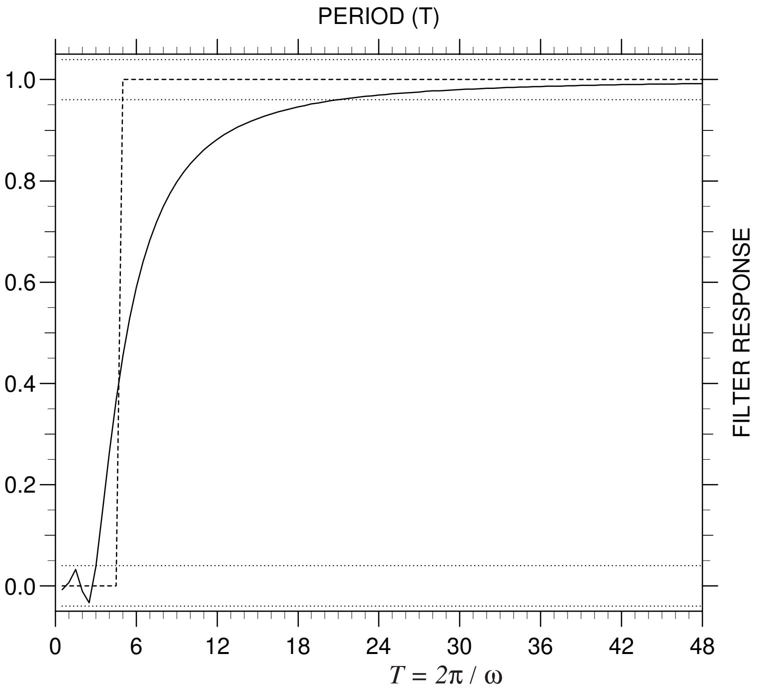
➤ Weights :

Minimizing the maximum error both :

- ♥ in the stop-band : $|H(\omega)| \le r$ if : $2\Delta t \le T \le T_s$ or $\omega_s \le \omega \le \pi / \Delta t$
- ♥ in the pass-band : $|H(\omega)\text{-}1| \le r$ if : $T \ge T_p$ or $0 \le \omega \le \omega_p$

➤ Equivalent cut-off period : $T_c = 2\pi / \omega_c$, $\omega_c = ½ (\omega_s + \omega_p)$

➤ Example :



LEGEND :

Full line : effective response

Dashed line : theorical response

Dotted lines : limit values
in the stop-band (below T_s)
and pass-band (above T_p)

FILTER CHARACTERISTICS :

Stop-band edge period : $T_s = 3$ h
Pass-band edge period : $T_p = 15$ h
Cut-off period : $T_c = 5$ h
Maximum deviation : r =.04
Time-span : $\mathcal{T}$ = 4 h
Time-step : Δt = 450 s
Order : $2N + 1 = \mathcal{T} / \Delta t + 1 = 33$

The "Dolph-Chebyshev" filter

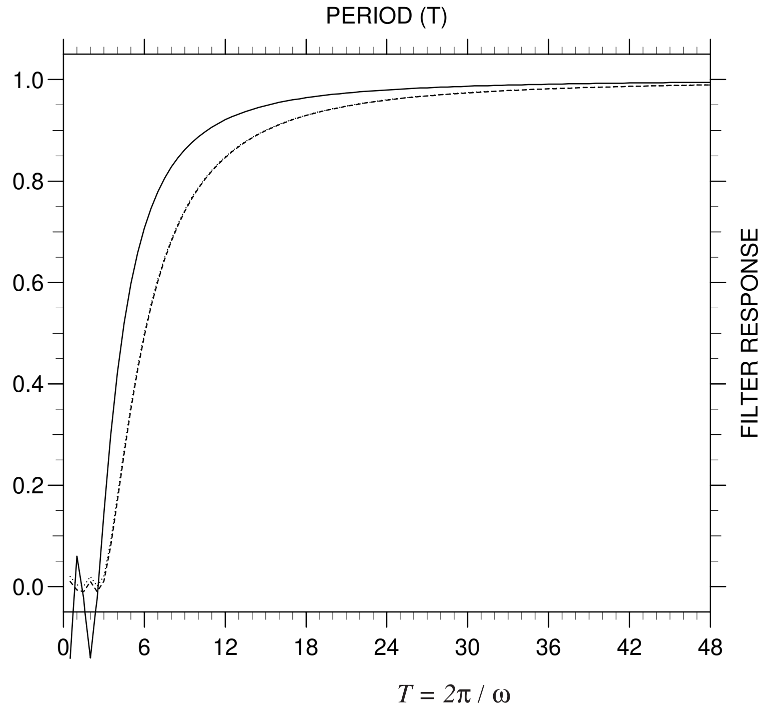
➤ Weights, for a chosen stop-band edge period T_s :

$$h_n = \frac{1}{2N+1}\left[1 + 2r\sum_{k=1}^{N} T_{2N}\left(\cos(\frac{\theta_k}{2}) / \cos(\frac{\theta_s}{2})\right) \cdot \cos(k\theta_n)\right], \quad \theta_n = 2\pi\frac{n}{N}, \quad \theta_s = 2\pi\frac{\Delta t}{T_s}, \; -N \le n \le N$$

➤ Notations

T_M : Chebyshev polynomial of order M
r : ripple-ratio (maximum deviation in the stop-band)
$2N$+1 : order of the filter

➤ Example :



LEGEND :

Full line : effective response (first filter)

Dashed line : theorical response (second filter)

Dotted line : squared response (first filter)

FILTERS CHARACTERISTICS :

Stop-band edge period : $T_s = 3\,\mathrm{h}$
Time-step : $\Delta t = 450\,\mathrm{s}$
Time-spans : $\mathcal{T}_1 = 2.5\,\mathrm{h}$, $\mathcal{T}_2 = 5.0\,\mathrm{h}$
Maximum deviations :
$r_1 = .14$, $r_2 = .01$, $r_1^2 = .02$

Orders :

$$2N_1 + 1 = \mathcal{T}_1 / \Delta t + 1 = 21$$
$$2N_2 + 1 = \mathcal{T}_2 / \Delta t + 1 = 41$$

Some more details about the "Dolph-Chebyshev" filter ...

➤ Chebyshev polynomials

$$T_M(x) = \begin{cases} \cos(M\cos^{-1}(x), & |x| < 1 \\ ch(M ch^{-1}(x)), & |x| > 1 \end{cases} \quad , T_0(x) = 1, \; T_1(x) = x, \; ..., \; T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \;\; n \geq 2$$

- ♦ $T_M(x)$ has M zeros in] -1 , +1 [and $(M+1)$ extrema in [-1 , +1]
- ♦ $|T_M(x)| \leq 1$ if $|x| \leq 1$ $\quad T_M(x) > 1$ if $x > 1$
- ♦ $T_M(x) \approx 2^{M-1} x^M$ if $x \to +\infty$

➤ Response function

$$H(\omega) = T_{2N}\left(\cos(\frac{\omega\Delta t}{2}) \Big/ \cos(\frac{\omega_s \Delta t}{2})\right) \Bigg/ T_{2N}\left(1 \Big/ \cos(\frac{\omega_s \Delta t}{2})\right), \qquad \omega_s = 2\pi / T_s, \quad \omega = 2\pi / T$$

➤ Maximum deviation in the stop-band (ripple-ratio)

$$r = T_{2N}\left(1 \Big/ \cos(\frac{\omega_s \Delta t}{2})\right)^{-1}, \;\; \omega_s = 2\pi / T_s$$

➤ Minimum time-span for given stop-band edge period and ripple-ratio

$$\mathcal{T}_{\min} = 2N_{\min}\Delta t \ , \quad 2N_{\min} \approx \frac{ch^{-1}\left(1/r\right)}{ch^{-1}\left(1/\cos\left(\frac{\omega_s \Delta t}{2}\right)\right)} \ , \quad \omega_s = 2\pi / T_s \qquad (N_{\min} \text{ integer})$$

$$\mathcal{T}_{\min} \approx Ts \ . \ ch^{-1}(1/r) \ / \ \pi \quad \text{as far as } 2\Delta t << Ts$$

The Dolph-Chebyshev filter is an optimal filter with the stop-band edge defined by : $H(\omega_p) = 1 - r$

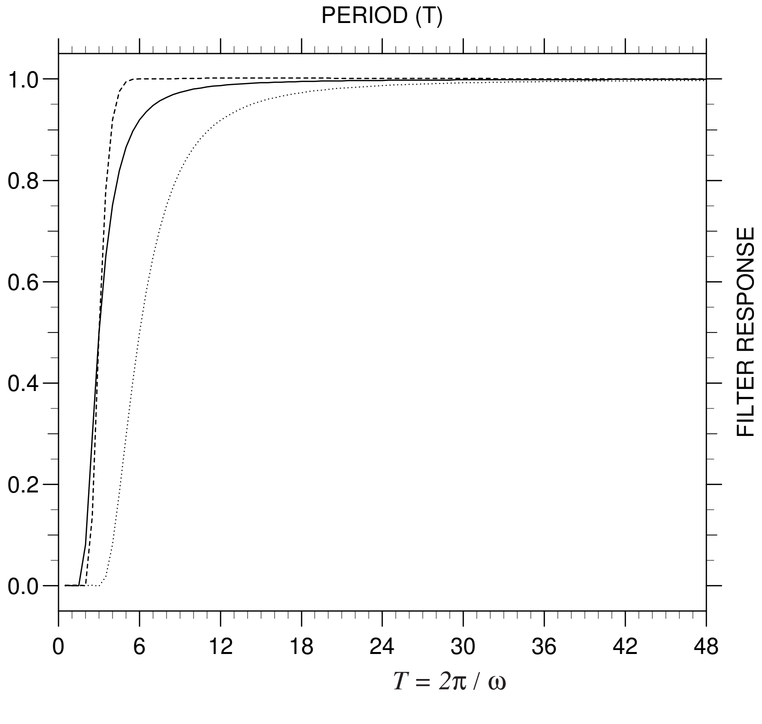
The "Ideal low-pass" filter with a "Dolph" window

➤ Weights, for a chosen cut-off period Tc :

$$h_n = \frac{\sin(n\omega_c \Delta t)}{n\pi} \times w_n \qquad , \ T_c = 2\pi / \omega_c \qquad , \ -N \leq n \leq N$$

(w_n) : weights of a Dolph-Chebyshev filter with the stop-band edge period equal to half the time-span.($Ts = N\, \Delta t = \mathcal{T} / 2$)

➤ Example :



LEGEND :

Full line : first filter

Dashed line : second filter

Dotted line : third filter

FILTERS CHARACTERISTICS :

Stop-band edge periods :
Ts = 3 h / 3 h / 6 h
Time-spans :
$\mathcal{T}$ = 6 h / 12 h / 12 h
Time-step : Δt = 900 s
Orders : $2N + 1 = \mathcal{T} / \Delta t + 1$
= 25 / 49 / 49