A Lower Boundary Condition for Nonhydrostatic ALADIN

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1 Introduction

Nonhydrostatic effects become important in atmospheric modelling at scales of typically 10km or lower. Nonhydrostatic effects in flow over orography on these scales can have a large impact on the quality of the resulting forecast. For instance, the distribution of rain-fall may be sensitive to a correct prediction of the flow field around and over orography.

The continuing increase of computing power available to meteorologists is allowing the possibility of representing nonhydrostatic dynamics in forecast models. Current operational versions of the ALADIN limited area forecast model employ a hydrostatic representation of the atmosphere. In order to allow this model to develop and fully take advantage of increases in model resolution and improvements in data assimilation, it is essential that the dynamical core should be rewritten to account for nonhydrostatic effects in the atmosphere. It has been successfully demonstrated that a nonhydrostatic version of ALADIN, known as ALADIN-NH, may be obtained by building on the foundation of the the existing hydrostatic model [2]. This model is based on the use of hydrostatic pressure as a vertical coordinate [3]. Many idealised tests have been successfully performed with ALADIN-NH. However, the model does not yet possess the necessary robustness for it to be used as an operational forecast model. Recent studies, therefore, have concentrated on issues of numerical stability for NH models such as ALADIN-NH [1]. It has been found that model stability is sensitive to the way in which vertical momentum is represented in the model.

2 ALADIN-NH Equation Sets

ALADIN-NH is formulated using the familiar primitive variables for temperature and horizontal wind. This is not the case for pressure and vertical wind, where derived variables are used as prognostic quantities. In fact whole families of prognostic variables have been formulated for these two quantities. Linear stability analysis shows that some combinations of prognostic variables offer greatly enhanced model stability, compared with the original choice of prognostic variables.

In this report we are only concerned with the vertical momentum equation. For inviscid nonhydrostatic fluid flow, the conservation of vertical momentum leads to the following prognostic equation,

$$\frac{\partial w}{\partial t} + \mathbf{V}.\nabla w + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{1}$$

This equation uses the primitive variables $\mathbf{V} = (u, v)$, for horizontal wind; p and ρ for pressure and density, respectively; and w for vertical wind.

ALADIN-NH uses a semi-implicit discretisation of the prognostic equations, which requires a reference atmosphere. The hydrostatic pressure of this reference atmosphere is denoted $\pi^\star = \pi^\star(\eta)$, where η is the generalised vertical coordinate defined implicitly by

$$\pi(t, \mathbf{x}, \eta) = A(\eta) + B(\eta)\pi_s(t, \mathbf{x}). \tag{2}$$

In (2) π is the Laprise hydrostatic pressure coordinate, A and B are prescribed functions, defining the vertical grid structure, and π_s is the hydrostatic surface pressure; we also use $\mathbf{x} = (x, y)$. Full details of the ALADIN-NH model may be found in [2].

For our purposes it is sufficient to know that the first working version of ALADIN-NH used modified prognostic variables to replace w and p,

$$P_0 = \frac{p - \pi}{\pi^\star} \tag{3}$$

$$d_0 = \frac{\partial(gw)}{\partial\phi^\star}. \tag{4}$$

The zero subscript indicates that these variables are the first in a subsequent family of variables. Additionally we have used the notation $\phi^\star = \phi^\star(\eta)$, this being the geopotential height of the reference atmosphere at coordinate level η, and g is the constant of gravitational acceleration.

Transforming the primitive wind equation (1) to the η coordinate system, and using the modified pressure variable P_0, we obtain

$$\frac{\partial(gw)}{\partial t} + \mathbf{V}.\nabla(gw) + \dot{\eta}\frac{\partial(gw)}{\partial\eta} = g^2\frac{\partial(\pi^\star P_0)}{\partial\pi}. \tag{5}$$

Since the choice of pressure variable plays no part in the present discussion, we shall from here on denote the nonhydrostatic pressure perturbation as $p - \pi$. Next, transforming from gw to d_0, we obtain

$$\frac{d(d_0)}{dt} = g^2\frac{\partial}{\partial\phi^\star}\frac{\partial(p-\pi)}{\partial\pi} - \frac{\partial\mathbf{V}}{\partial\phi^\star}.\nabla(gw) - d_0\frac{\partial}{\partial\phi^\star}\left(m\dot{\eta}\frac{\partial\phi^\star}{\partial\pi}\right), \tag{6}$$

where $m = \partial\pi/\partial\eta$ and we now use d/dt to denote the material derivative. If d_0 is used as a prognostic variable then gw, which appears in (6), must be calculated diagnostically using

$$gw(t, \mathbf{x}, \eta) = gw_s(t, \mathbf{x}) - \int_\eta^1 d_0 \frac{d\phi^\star}{d\eta}\, d\eta. \tag{7}$$

This last equation introduces w_s, which is the vertical wind at the lower boundary of the model. At this point we must consider the upper and lower boundary conditions and how they affect our choice of prognostic variable.

3 Upper and Lower Boundary Conditions

The dynamics kernel of a NWP model is generally designed to represent dry, inviscid dynamics. In this highly simplified model of the atmosphere the lower boundary condition is determined solely by the geometry of the underlying orography. The lower boundary is formed of a solid material surface. There is, therefore, no flow normal to the surface. Also, since the fluid is assumed to be inviscid, we know that it is incapable of supporting any shear stress. At a material surface this results in the so-called free-slip boundary condition. It is important to note that we are not free to choose this boundary condition arbitrarily: for an inviscid fluid the free-slip condition must be satisfied. In the context of the ALADIN model, the free-slip lower boundary condition is

$$gw_s(t, \mathbf{x}) = \mathbf{V}_s(t, \mathbf{x}).\nabla\phi_s(\mathbf{x}) \quad \forall t, \tag{8}$$

where subscript s denotes a value on the lower boundary (surface). In the discretised equations of ALADIN-NH the horizontal wind is not represented on the lower boundary. The discretised equations use the horizontal wind on the first interior grid-level, in place of $\mathbf{V}_s$, whenever the free-slip boundary condition is invoked.

For NWP applications the lower boundary condition is of much greater importance than the upper boundary condition. This is due to the fact that surface processes are generally of primary importance in determining weather. It is far from clear what the best upper boundary condition is for NWP purposes. We mention three possibilities. The rigid lid upper boundary condition represents a solid material surface at the top of the model,

$$w(t, \mathbf{x}, \eta = 0) = 0 \quad \forall t. \tag{9}$$

Another alternative is the free-surface boundary condition,

$$gw_T = \frac{\partial\phi_T}{\partial t} + \mathbf{V}_T.\nabla\phi_T, \tag{10}$$

where subscript T denotes values on the top boundary.

How should the upper and lower boundary conditions be applied when d_0 is used as prognostic variable? By integrating through the depth of the atmosphere, we see that d_0 must satisfy the integral constraint

$$\int_0^1 d_0 \frac{d\phi^\star}{d\eta}\, d\eta = g w_s - g w_T. \tag{11}$$

An upper boundary condition for d_0 can be obtained by making use of the continuity equation,

$$\frac{\partial m}{\partial t} + \nabla.(m\mathbf{V}) + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0, \tag{12}$$

where

$$m = \frac{\partial \pi}{\partial \eta}. \tag{13}$$

The approach of applying upper and lower boundary conditions directly to d_0, as outlined here, is not used in ALADIN-NH. It is not necessary to use these conditions in order to close the discretised system of equations. This is fortunate, as applying these conditions numerically would be a technically extremely difficult. In ALADIN-NH, the upper and lower boundary conditions for pressure serve to close the system, in place of boundary conditions on d_0.

4 Numerics

In order to test the lower boundary condition of the ALADIN-NH model, a number of idealised tests have been made involving flow over orography. These tests all involve the calculation of a steady-state solution. These are obtained by careful initialisation of the model then running the model until a quasi-steady state is reached. If an Eulerian discretisation of the flow equations is used, then a steady state test is not a very severe test. If the computed solution begins sufficiently close to the steady-state, then Eulerian time derivatives will be small. The result of this is that numerical errors may be smaller than they would be in any realistic time dependent situation. Consequently, convergence to the correct solution is unproblematic. The same argument does not apply to the semi-Lagrangian (SL) scheme. Even once a steady-state solution has been reached, a SL scheme must continue to compute trajectories. Since time integration is performed along these trajectories there may still be significant errors due to temporal differencing, even at steady-state. In view of this, we shall restrict our attention to steady-state tests with the SL version of ALADIN-NH.

ALADIN-NH allows a variety of different temporal discretisations to be used. All of these involve the use of the semi-implicit method to control fast propagating modes of the solution. We shall consider just one of these schemes, the two-time-level scheme

$$X^+ - \frac{\triangle t}{2}L^+ = X_o^0 + \frac{1}{2}\left(M^0 + M_o^0\right) - \frac{\triangle t}{2}L^0. \tag{14}$$

In this schematic, X represents the state vector of the solution, M is the material time-derivative of X; the linear function L is obtained by linearising M about some reference state; superscript $+$ denotes time $t + \triangle t$, and 0 denotes time t; subscript o indicates interpolation to departure points.

Applying this scheme to the prognostic variable d_0, we obtain

$$d^+ - \frac{\triangle t}{2}L_d^+ = d_o^0 + \frac{1}{2}\left[M_d^0 + \left(M_d^0\right)_o\right] - \frac{\triangle t}{2}L_d^0. \tag{15}$$

In this equation we have dropped the subscript from d_0 for clarity. The nonlinear model, M_d, is given by the right-hand side of (6). This right-hand side contains terms which derive from the mapping between w and d_0: the mapping from w to d_0 has been used in order to obtain (6) from (5). This means that M_d must account for the dependency on spatial position, of the w to d_0 transform, along a trajectory. This is in strong contrast to the treatment of the horizontal wind in ALADIN-NH. In the case of the horizontal wind, horizontal divergence is used in the semi-implicit stage. However, the first approximation to the solution at time $t+\triangle t$ is obtained by discretising the differential equations for V, rather than the equations for horizontal divergence and vorticity.

An additional consequence of using d_0 as a prognostic variable is that the discrete equation (15) requires the calculation of vertical summations. These summations allow any errors introduced near the lower boundary to be immediately propagated upwards through the depth of the model. In addition to problems of error propagation there may also be computational stability issues. Since the summation is made from the lower boundary upwards, downstream information is being used in the discretisation when the wind is directed downwards. Downstream discretisations may fail to converge to the analytic domain of dependence when applied to hyperbolic equations.

An alternative approach, which avoids some of these problems, is to discretise the vertical wind equation (5), to give

$$gw^+ - \frac{\triangle t}{2}L_w^+ = gw_o^0 + \frac{\triangle t}{2}\left[M_w^0 + \left(M_w^0\right)_o\right] - \frac{\triangle t}{2}L_w^0. \tag{16}$$

The nonlinear function, M_w, is given by the right-hand side of (5) and L_w is its linearisation. We see that M_w has fewer terms than M_d. As a consequence, the error in representing M_w, using finite differences, is likely to be smaller than the corresponding error introduced by discretising M_d.

The scheme (16) does not automatically satisfy the free-slip lower boundary condition. However, once $\mathbf{V}^+$ has been calculated, w_s^+ may be calculated using the free-slip condition (8). This diagnosed value, of the vertical wind on the lower boundary, is used in place of the solution on the lower boundary given by (16). In this way we are able to approximate the lower boundary condition more closely than was the case with d_0. In the case of d_0 the free-slip boundary condition is not applied directly to the d_0 field, for reasons of computational complexity, as mentioned earlier.

In order to implement this scheme it is necessary to separate the semi-implicit scheme (14) into two stages: first, an explicit approximation of the solution at time $t + \triangle t$,

$$\widetilde{X}^{+} = X_o^0 + \frac{1}{2}\left(M^0 + M_o^0\right); \tag{17}$$

and, secondly, the semi-implicit filter stage,

$$X^{+} - \frac{\triangle t}{2} L^{+} = \widetilde{X}^{+} - \frac{\triangle t}{2} L^{0}. \tag{18}$$

Once explicit approximations have been made for all model fields, the approximation for vertical wind is converted to an approximation for the vertical derivative,

$$\tilde{d}^{+} = \frac{\partial}{\partial \phi^{\star}}(\widetilde{gw}^{+}). \tag{19}$$

The filtered solution is then given by solving the following filter equation for d_0^{+}:

$$d^{+} - \frac{\triangle t}{2} L_d^{+} = \tilde{d}^{+} - \frac{\triangle t}{2} L_d^{-}. \tag{20}$$

Finally, the wind at time $t + \triangle t$ is recovered by an inversion of (19),

$$gw^{+} = \mathbf{V}_s.\nabla\phi_s - \int_{\eta}^{1} d_0^{+} \frac{d\phi^{\star}}{d\eta}\, d\eta. \tag{21}$$

This method for applying the free-slip boundary condition on the lower boundary is not restricted to the use of only d_0 in the semi-implicit filter stage. Other variables, such as $d_3 = \partial(gw)/\partial\phi$, may also be used. This is important, as the stability of the model using d_0 in the semi-implicit stage is insufficient for operational weather forecasting. Other choices of vertical derivative variable give much enhanced stability properties, as described in the report of Jozef Vívoda under this same Training Network.

The effect of using w in the explicit predictor stage, rather than d_0, may be seen in Figure 1. This shows steady-state solutions for a test of nonhydrostatic flow over orography. When d_0 is used as the predicted variable a spurious defect is present in the solution, associated with the orography. This defect disappears when w is used as the explicitly predicted variable in the way described here.

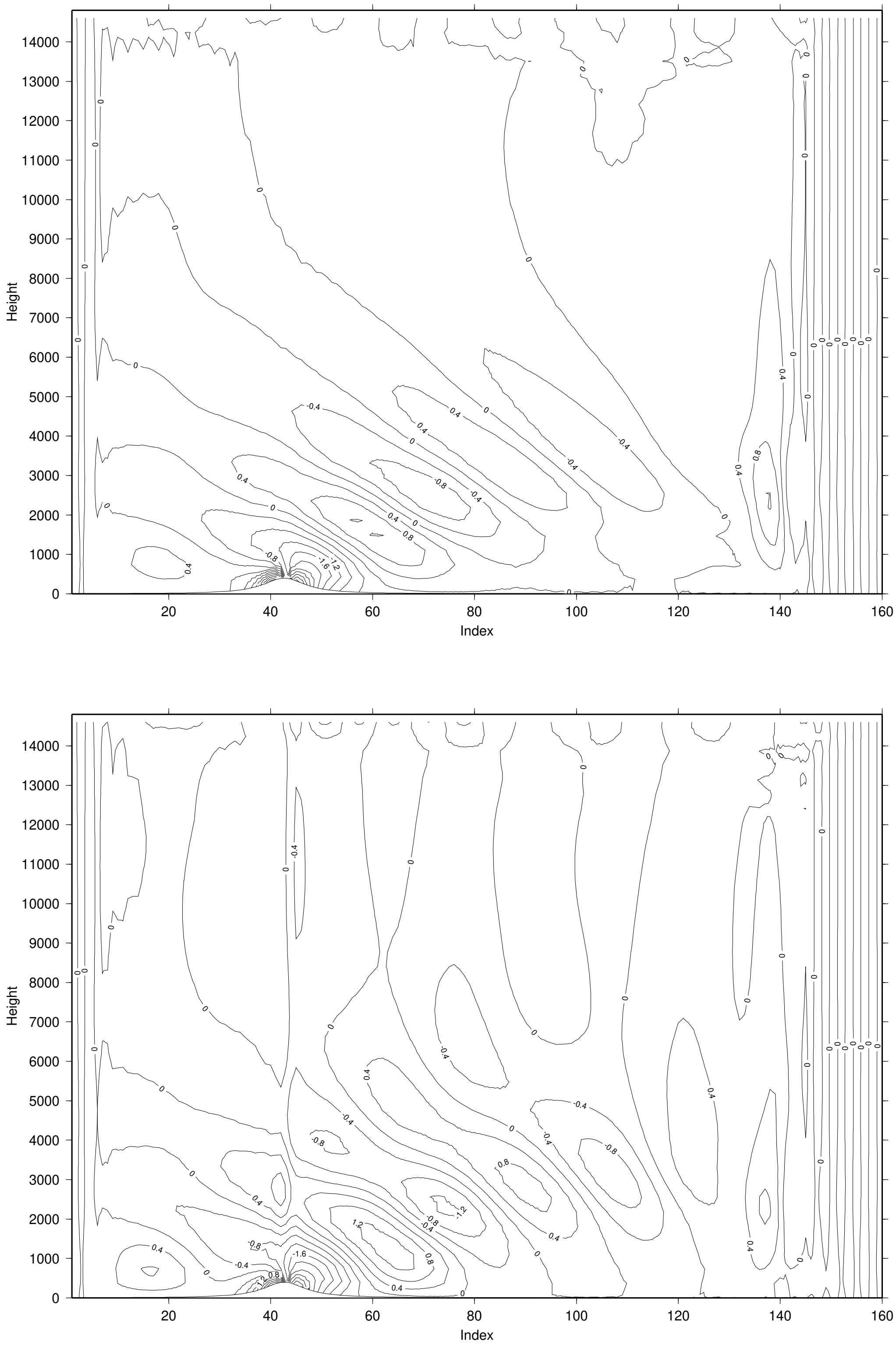

Figure 1: *Steady-state nonhydrostatic flow test. Vertical wind plotted. Solution below was computed using d_0 as prognostic variable; solution above using w.*

References

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