ALADIN NH:
First results with Vertical Finite Elements Discretisation

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Introduction

Motivation:

- VFE scheme successfully implemented into HY model (Untch and Hortal,2004)
- To extent the VFE scheme into NH model with HY model as a limit case
- VFE has two times higher accuracy than FD method on the same stencil (eight order with cubic elements in the case of evenly spaced mesh) – two times higher accuracy with the same amount of levels than FD
- More accurate vertical velocities for SL scheme

VFE basic features

Starting point – Untch and Hortal,2004:

- Basis functions – cubic B-splines with compact support
- No staggering – all variables are defined on model full levels
- Only integration/derivation is performed in FE space, the products of variables are done in physical space (derivation/integration is just matrix multiplication)
- In SL version of HY model (ECMWF or ARPEGE) only non-local operations in the vertical are integrations. In NH version derivatives plays crucial role.

FE derivative operator

We expand F and f in terms of chosen set of functions:

$$F(x,\eta) = \frac{\partial f(x,\eta)}{\partial \eta} \quad \Rightarrow \quad \sum_i \hat{F}(x)_i d(\eta)_i - \sum_i \hat{f}(x)_i \frac{\partial e(\eta)_i}{\partial \eta} = R$$

Truncation error R is required to vanish in weighted integral sense on model domain (0,1):

$$\int_0^1 R\,\Psi j = 0 \quad \Rightarrow \quad \sum_i \hat{F}_i \int_0^1 d_i \Psi_j d\eta = \sum_i \hat{f}_i \int_0^1 \frac{de_i}{d\eta} \Psi_j d\eta$$

$$A\hat{F} = B\hat{f}$$

Finally we incorporate the transforms from physical to FE space and back:

$$F = T\hat{F}$$

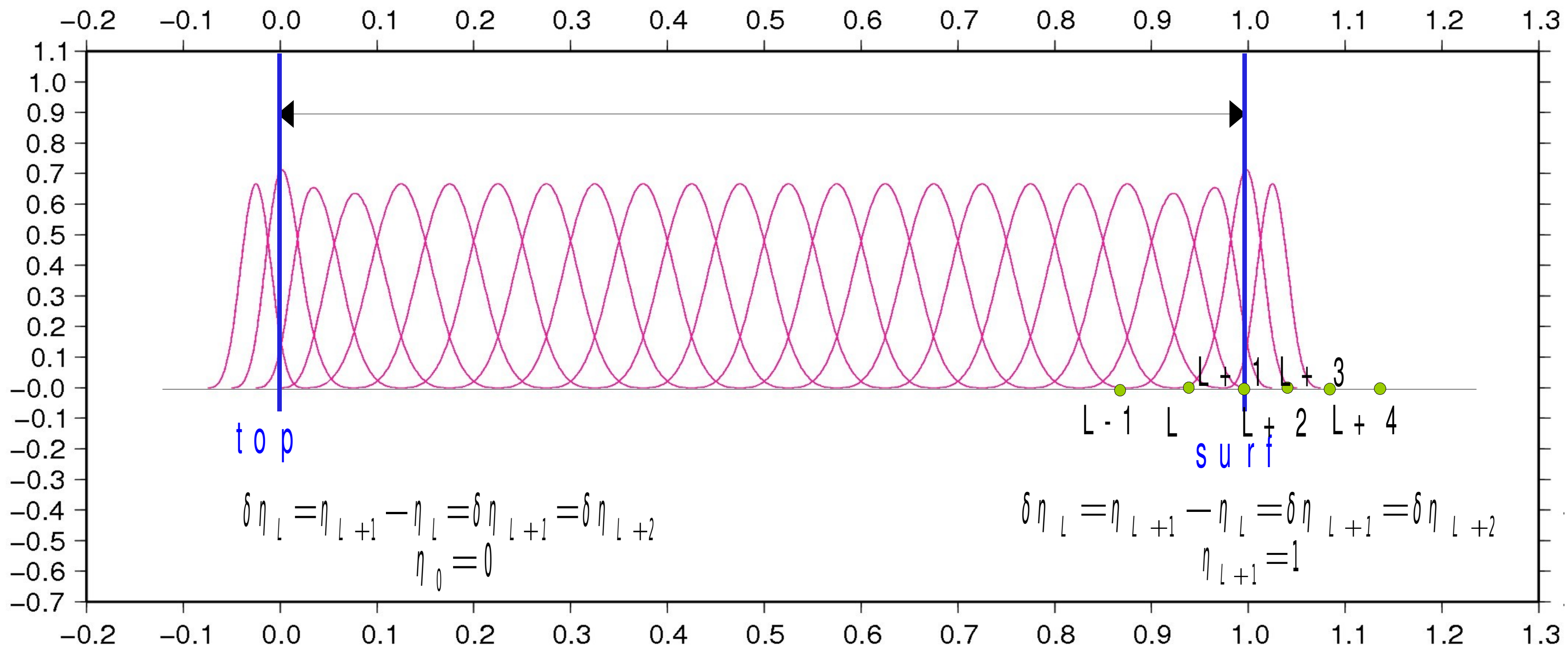
$$\hat{f} = C^{-1} f$$

$$\Rightarrow \quad F = \mathbf{TA}^{-1}\mathbf{BC}^{-1} f = \mathbf{Df}$$

FE derivative operator - basis functions (1)

$d(\eta)_i, e(\eta)_i, \psi(\eta)_i \quad i \in \{1, L+4\}$

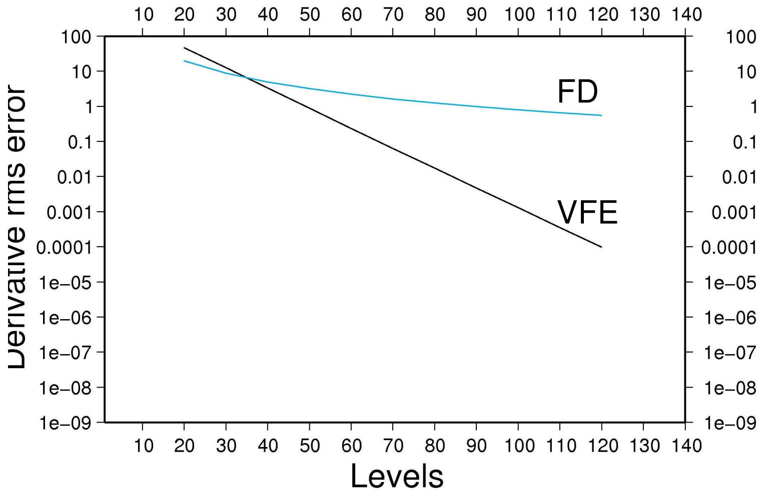
Example for 35 unevenly spaced levels



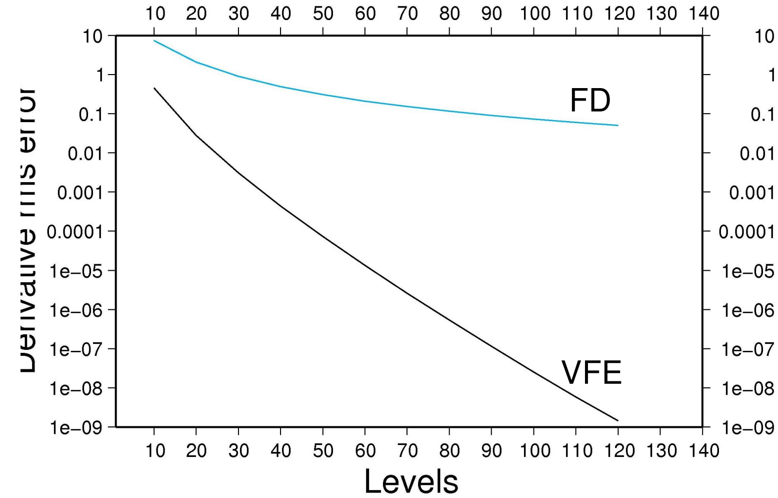
- In order to cover the physical domain (eta=<0,1>) with the full set of basis functions the 4 additional functions must be determined, the two at the bottom and the two at the top (analogical approach as Untch and Hortal, 2004).

FE derivative operator accuracy

Test function: $f(\eta) = \sin(6\pi\eta)$



Dependence of accuracy on chosen BCs



The 4 additional assumptions:

$$f(\eta=0) = f_1$$

$$f'(\eta=0) = 0$$

$$f(\eta=1) = f_L$$

$$f'(\eta=1) = 0$$

$$f(\eta=0) = \frac{\delta\eta_1 - \delta\eta_0}{\delta\eta_1} f_1 - \frac{\delta\eta_0}{\delta\eta_1} f_2$$

$$f'(\eta=0) = \frac{f_1 - f_2}{\delta\eta_1}$$

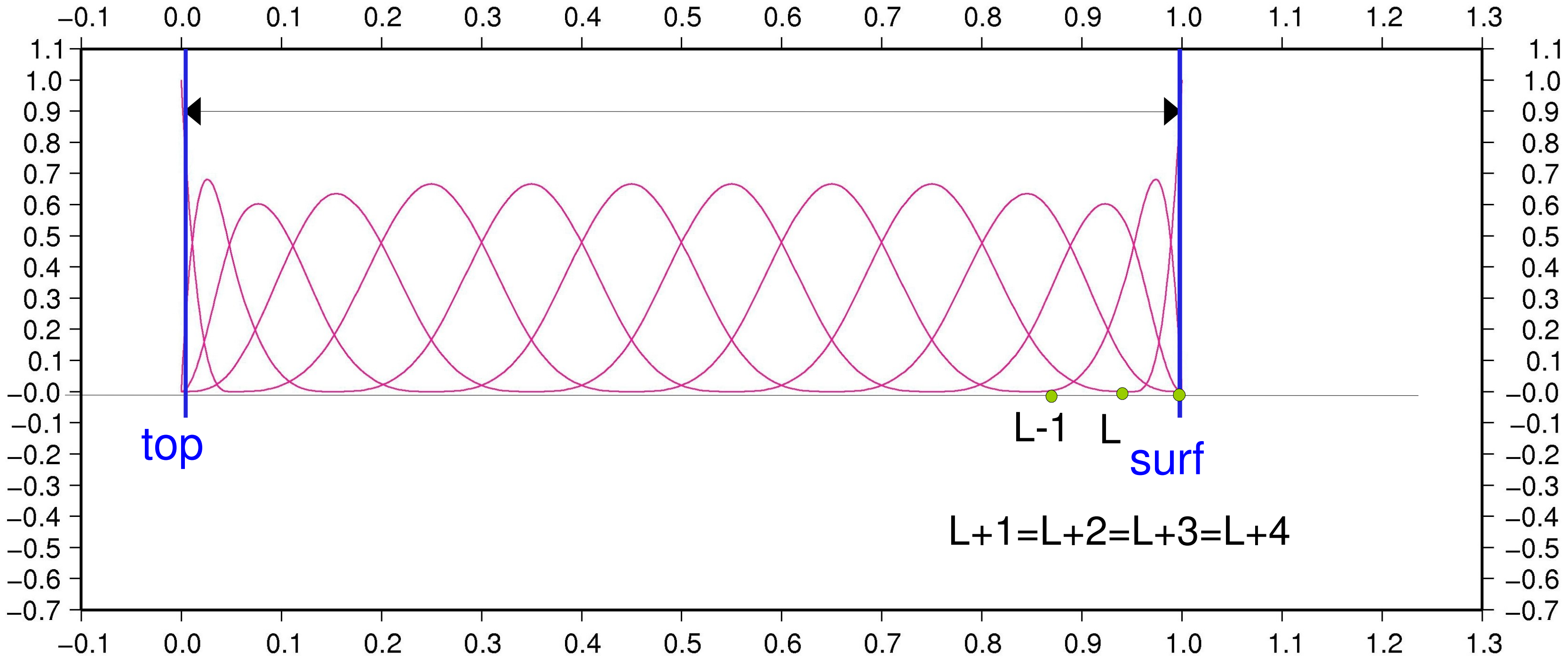
$$f(\eta=1) = \frac{\delta\eta_{L-1} - \delta\eta_L}{\delta\eta_{L-1}} f_L - \frac{\delta\eta_L}{\delta\eta_{L-1}} f_{L-1}$$

$$f'(\eta=1) = \frac{f_L - f_{L-1}}{\delta\eta_{L-1}}$$

FE derivative operator - basis functions (2)

$d(\eta)_i, e(\eta)_i, \psi(\eta)_i \quad i \in \{1, L+2\}$

3rd order multiplicity of nodes at the boundaries



- B-splines computed using De Boor (1978) recursive algorithm
- Only two additional conditions required (one at the bottom, one at the model top)
- Linear model unstable with this set of basis functions
- Tested with the non-linear terms (shown later in this presentation)

VFE scheme – linear NH model

System describing small amplitude waves around reference state $\bar{X}=(\bar{T},\bar{\pi})$

Prognostic var.

$$\hat{q}=\ln\left(\frac{p}{\pi}\right)$$

$$d_4=\frac{p}{mRT}\vec{\nabla}\Phi\frac{\partial\vec{v}}{\partial\eta}-\frac{gp}{mRT}\frac{\partial w}{\partial\eta}$$

$$q_s=\ln(\pi_s)$$

$$D=\vec{\nabla}\vec{v}$$

Linear system

$$\frac{\partial D}{\partial t}=-R\bar{G}\,T+R\bar{T}(\bar{G}-I)\,\hat{q}-R\bar{T}\,\hat{q}_s-\Delta\Phi_s$$

$$\frac{\partial T}{\partial t}=-\frac{R\bar{T}}{C_v}(D+d)$$

$$\frac{\partial q_s}{\partial t}=-\bar{N}D$$

$$\frac{\partial d}{\partial t}=-\frac{g^2}{R\bar{T}}\bar{L}\hat{q}$$

$$\frac{\partial\hat{q}}{\partial t}=-\frac{C_p}{C_v}(D+d)+\bar{S}D$$

Vertical operators

$$(\bar{G}X)_\eta=\int_\eta^1\frac{1}{\bar{\pi}}\frac{d\bar{\pi}}{d\eta'}X\,d\eta'$$

$$(\bar{N}X)_\eta=\frac{1}{\bar{\pi}_s}\int_0^1\frac{d\bar{\pi}}{d\eta'}X\,d\eta'$$

$$(\bar{S}X)_\eta=\frac{1}{\bar{\pi}}\int_0^\eta\frac{d\bar{\pi}}{d\eta'}X\,d\eta'$$

$$(\bar{L}X)_\eta=\frac{\bar{\pi}}{\bar{m}}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{m}}\frac{\partial\bar{\pi}X}{\partial\eta}\right)$$

Blue box: Mass weighted vertical integral operators (already in HY model)

Red box: Mass weighted vertical laplacian operator (NH specific)

Constraint C1 on vertical operators

•When eliminating the linear system to obtain single variable Helmholtz solver the following two equations for (d4,D) are obtained:

$$\left[I-\tau^2\left(c^{¿^2}-RT^{¿}C1\right)m^{¿2}\right]D=D+\tau^2\left(-RT^{¿}G^{¿}+c^{¿^2}\right)d_4$$

$$\left[I-\tau^2\frac{c^{¿^2}}{rH^{¿2}}L^{¿}\right]d_4=d_4+\tau^2\frac{1}{rH^{¿2}}L^{¿}\left(-RT^{¿}S^{¿}+c^{¿2}\right)m_{¿}^2D$$

Constrain C1 is 0 in continuous case: $C1: G^{¿}S^{¿}-S^{¿}-G^{¿}+N^{¿}=0$

•In discrete case C1 is NOT automatically 0. The FD vertical discretization is designed in order to satisfy C1. If C1 is not satisfied further elimination is not feasible.

•In order to be consistent with existing HY and NH code the C1 constrain shall be satisfied.

•C1 is not satisfied with VFE => 2Lx2L solver must be solved in spectral space or iterative solver with already existing solver must be applied (loosing the property of direct solver)

Constraint C2 on vertical operators

When C1 is assumed to be satisfied we could eliminate D from the system. It gives the structure equation

$$\left[I - \tau^2 c^{¿^2} \left(\;+ \frac{L^{¿}}{rH^{¿2}} \right) - \tau^4 \frac{N^{¿2} c^{¿^2}}{r} \quad ¿ \right] d_4 = d_4$$

With C2 constraint: $$C2: g^2 L^{¿} \left(S^{¿} G^{¿} - \frac{c_p}{c_V} S^{¿} - \frac{c_p}{c_V} G^{¿} \right) = N^{¿^2} c^{¿2} T^{¿}$$

•In continuous case **T*** is an identity matrix. In FD case **T*** is the tridiagonal matrix.

•For stability reasons of SI time stepping algorithm we require:

- L* to have real and negative eigenvalues
- **T*** to have positive and real eigenvalues

•In the case of C1 unsatisfied above conditions could be insufficient, we still require them assuming that C1 is close to zero

Laplacian term FE treatment

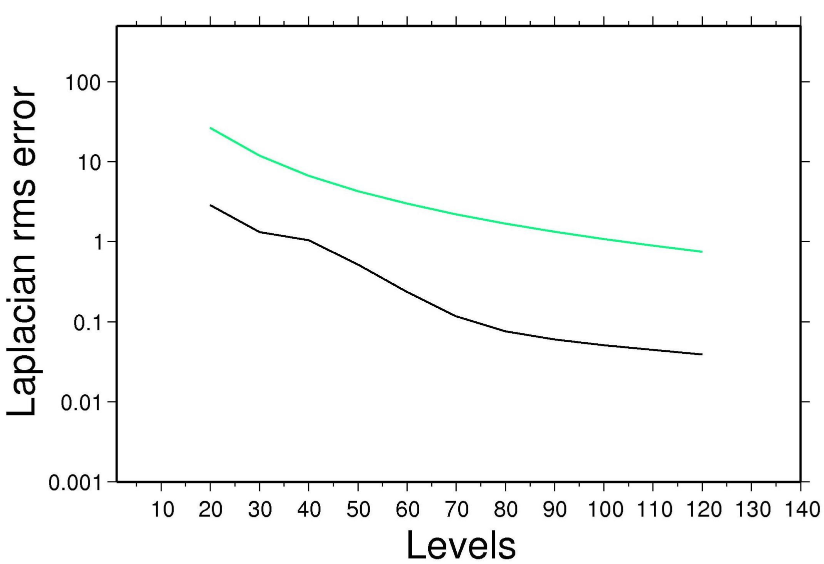
In linear and nonlinear model the laplacian is explicit and has the same form (thanks to fact that we use vertical divergence related prognostic variable):

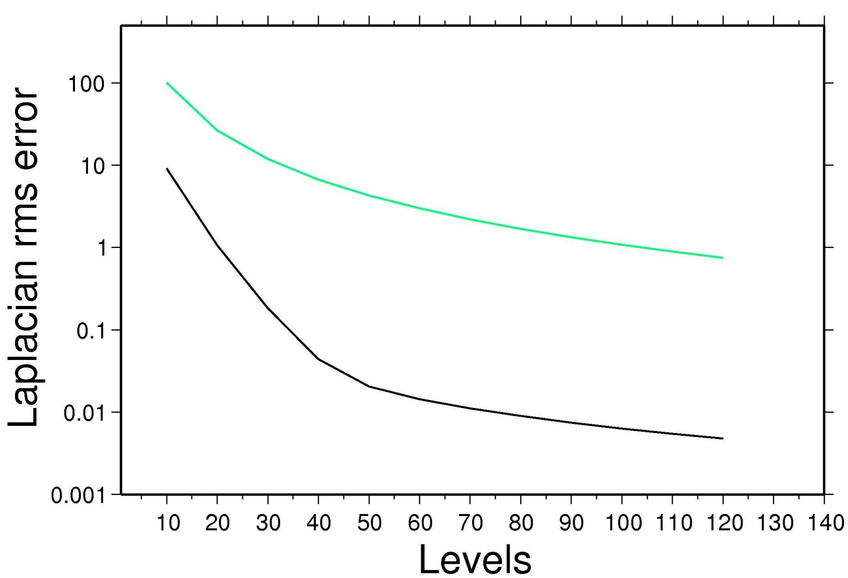
V1: $$LP=\frac{\pi}{m}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial \pi P}{\partial\eta}\right)$$

V2: $$LP=\frac{1}{m}\frac{\partial}{\partial\eta}\left(\frac{\pi^2}{m}\right)\frac{\partial P}{\partial\eta}+\left(\frac{\pi}{m}\right)^2\frac{\partial}{\partial\eta}\left(\frac{\partial P}{\partial\eta}\right)$$

- Boundary conditions the same as in FD model version

$$f(\eta)=\sin(6\pi\sigma)$$





Laplacian term FE treatment

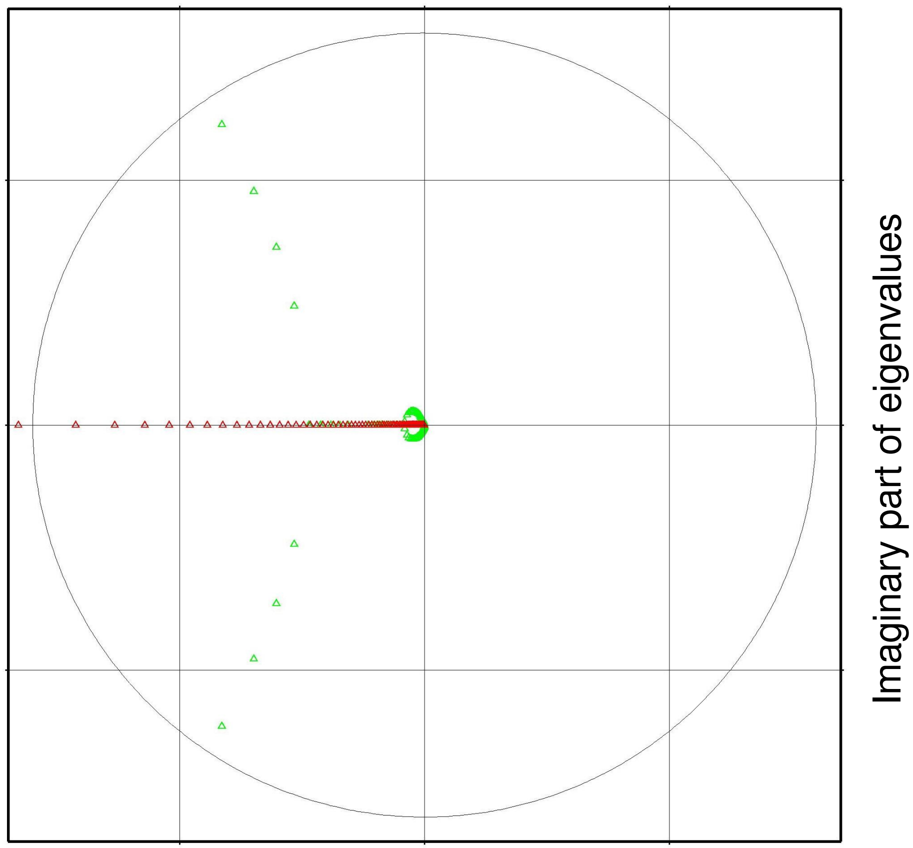
V1: $$LP = \frac{\pi}{m}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial \pi P}{\partial\eta}\right)$$

V2: $$LP = \frac{1}{m}\frac{\partial}{\partial\eta}\left(\frac{\pi^2}{m}\right)\frac{\partial P}{\partial\eta} + \left(\frac{\pi}{m}\right)^2\frac{\partial}{\partial\eta}\left(\frac{\partial P}{\partial\eta}\right)$$

Eigenvalues of laplacian for 100 levels

Green – eigenvalues of V1
Red – eigenvalues of V2

Imaginary part multiplied by 100



Stability of 2TL Non-Extrapolating Iterative Centered Implicit scheme

Predictor:

$$\frac{X_F^{+(0)} - X_O^t}{\delta t} = \frac{R(X^t)_F + R(X^t)_O}{2} + B^{¿} . \frac{X_F^{+(0)} + X_O^t}{2}$$

Nth-corrector:

$$\frac{X_F^{+(n)} - X_O^t}{\delta t} = \frac{R(X^{+(n-1)})_F + R(X^t)_O}{2} + B^{¿} . \frac{X_F^{+(n)} + X_O^t}{2}$$

For the purpose of analysis of stability we linearise the nonlinear residual R around resting, isothermal, hydrostatically balanced reference state:

$$R(X) = M(X) - B^{¿} . X \approx \bar{B} . X - B^{¿} . X$$

We analyse the temperature instability due to fact that $T^{¿} \neq \bar{T}$

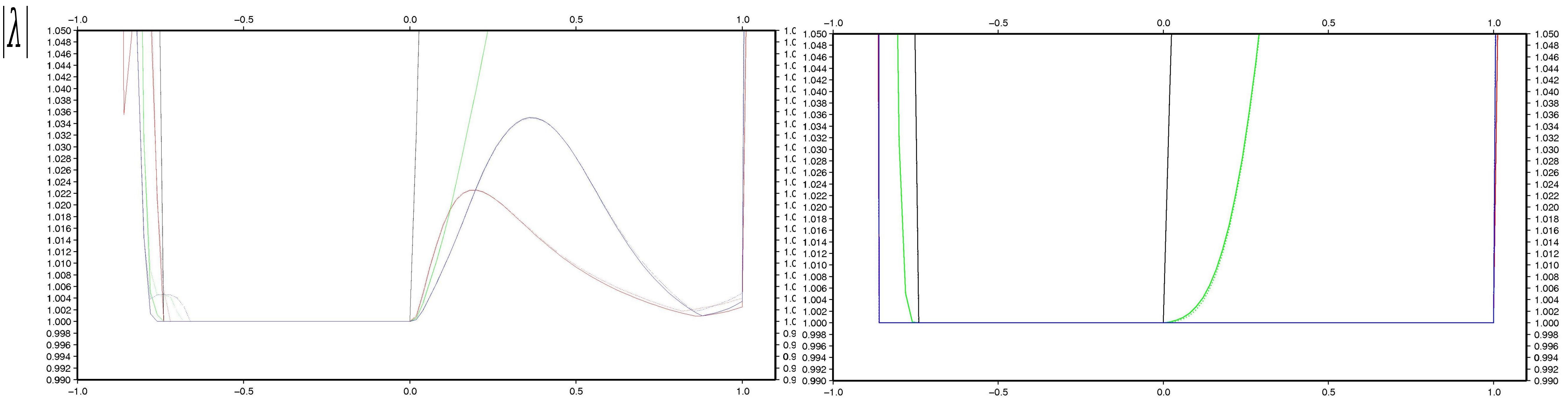
The following analyses were computed for the wave with dx=5km, dt=600s, 35 vertical levels, T*=350K, T*a=100K, p*s=101325Pa (perfect pressure)

C1 constrain unsatisfied – 2L solver stability

Stability properties ate equivalent to those of FD model

Dt=600s

Dt=999999s



$$\frac{\bar{T}-T^{¿}}{T^{¿}}$$

SI N=1 N=2 N=3 Solid – VFE Dotted - FD

$$\frac{\bar{T}-T^{¿}}{T^{¿}}$$

VFE scheme – iterative spectral solver

- Large horizontal domains requires the variable map factor to be considered in spectral space
- Effective implementation possible only when the single variable solver is factorized and the vertical and horizontal operators are separated

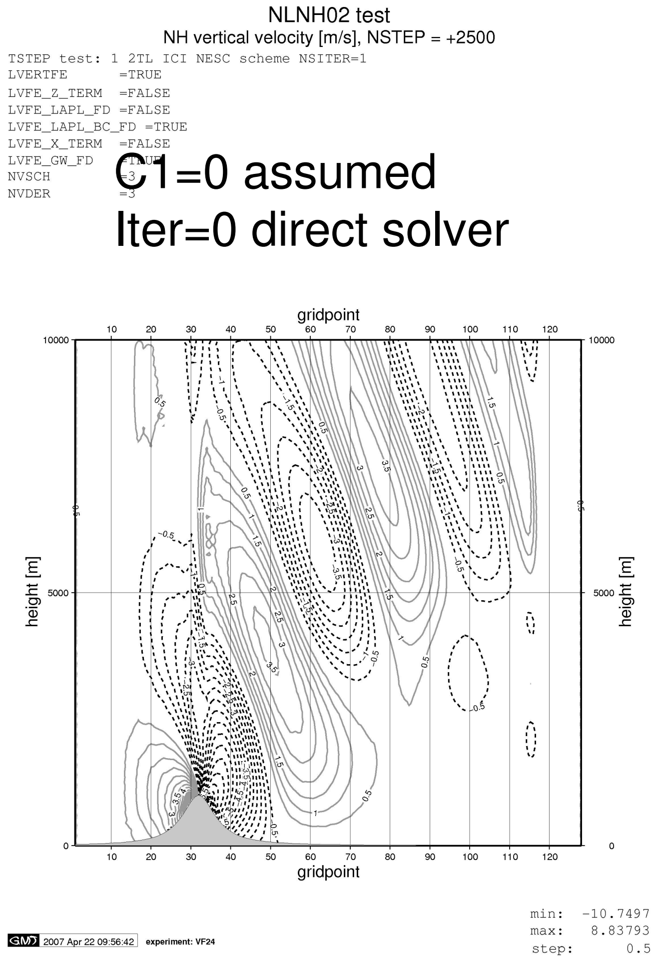
$$\left[I-\tau^2 B_V \left(\bar{M}\ \right)_H\right] d^+ = \hat{d}$$

- Separation of spatial operators is not tractable when 2L solver with nonzero C1 is applied
- We therefore implemented the iterative version (term with C1 moved to RHS and treated explicitly) with existing solver as a kernel

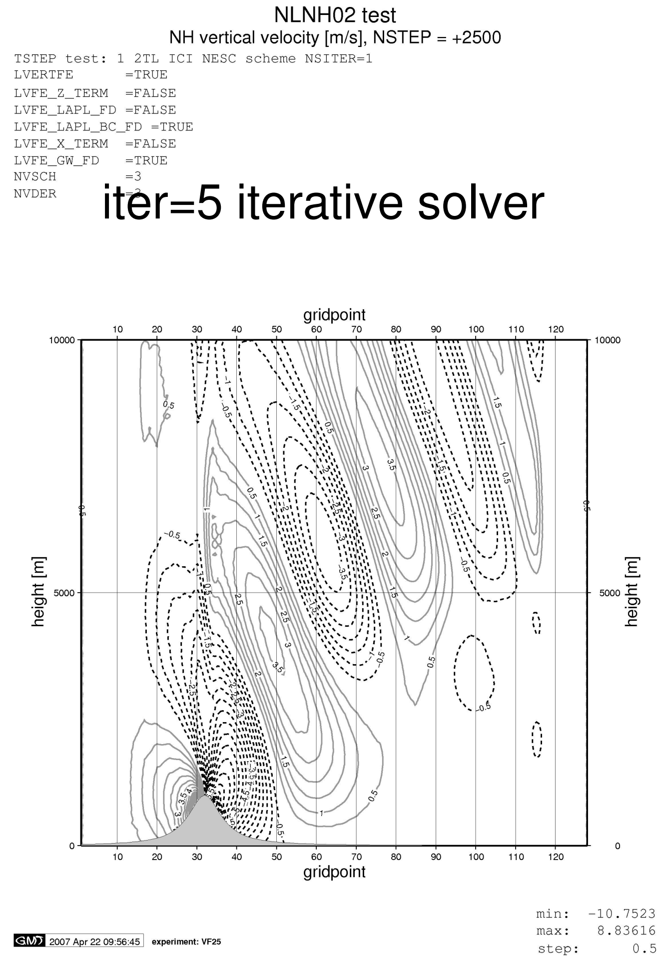
$$\left[I-\tau^2 B_V \left(\bar{M}\ \right)_H\right] d_n^+ = \hat{d} + corr\left(C1\left(D_{n-1}^+\right)\right)$$

VFE scheme – iterative solver

•Tested on NLNH flow in 2D framework, no effect of iterations visible, 30% CPU increase in adiabatic 2D framework, not yet tested in 3D



C1=0 assumed
Iter=0 direct solver



iter=5 iterative solver

VFE scheme – non-linear model discretisation

- CY30T1
- Tested in 2D
- 2TL ICI NESC n=1
- sponge

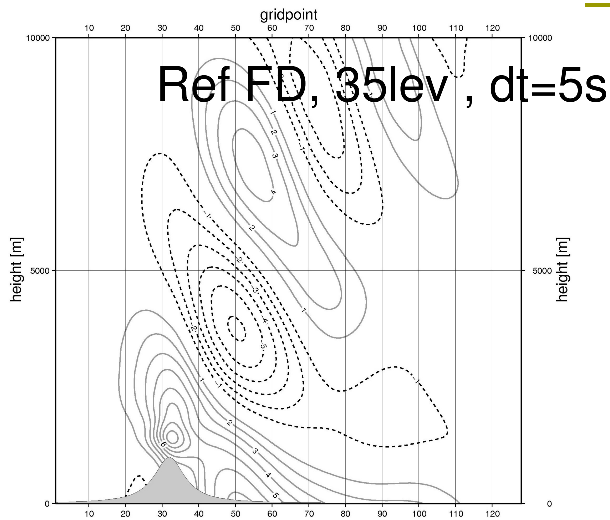
- Eta coordinate
- NLNH flow regime
- dx=200m
- V=10ms-1
- Agnesi hill, h=1000m, L=1000m

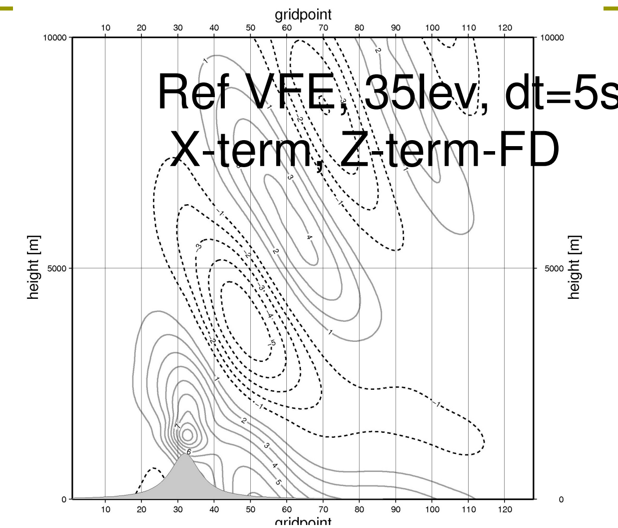
The results are sensitive to the discretization of non-linear terms (X-term and Z-term) contains vertical derivative of hor. wind

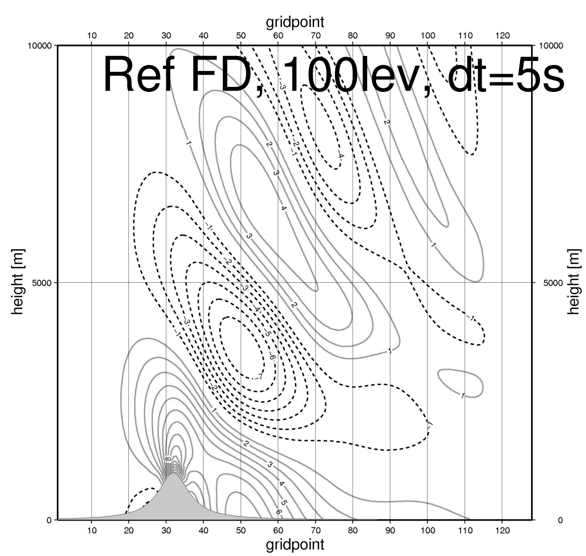
$$Z = \frac{p}{mRT} \nabla gw \frac{\partial \vec{v}}{\partial \eta}$$

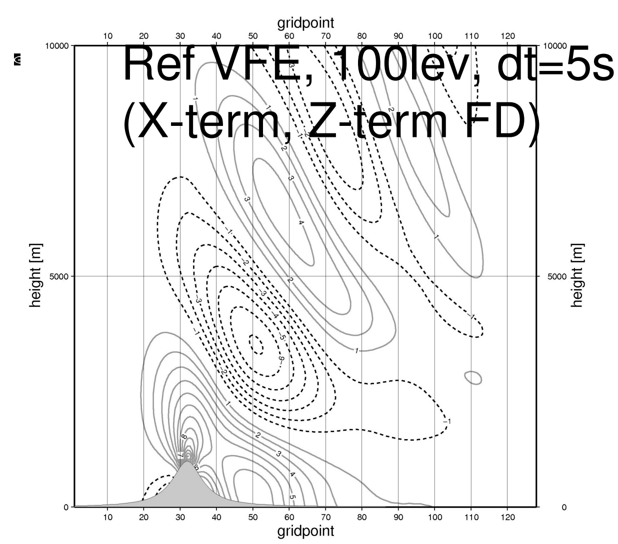
$$X = \frac{p}{mRT} \nabla \Phi \frac{\partial \vec{v}}{\partial \eta}$$

VFE scheme – comparison with FD scheme



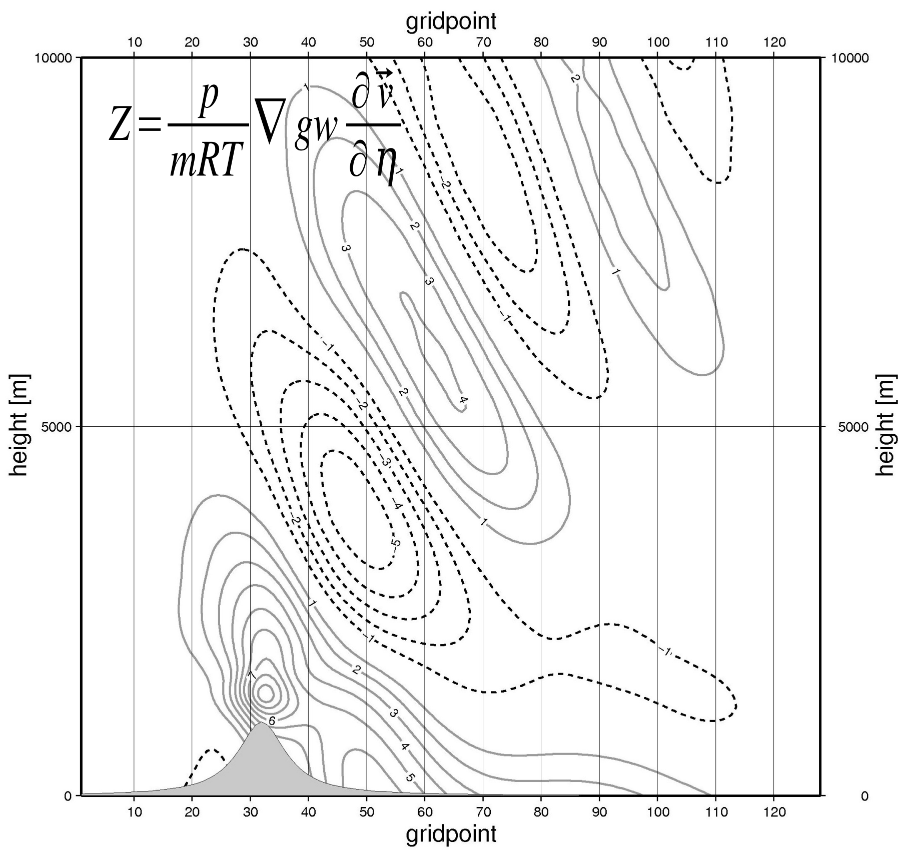






VFE scheme – non-linear terms

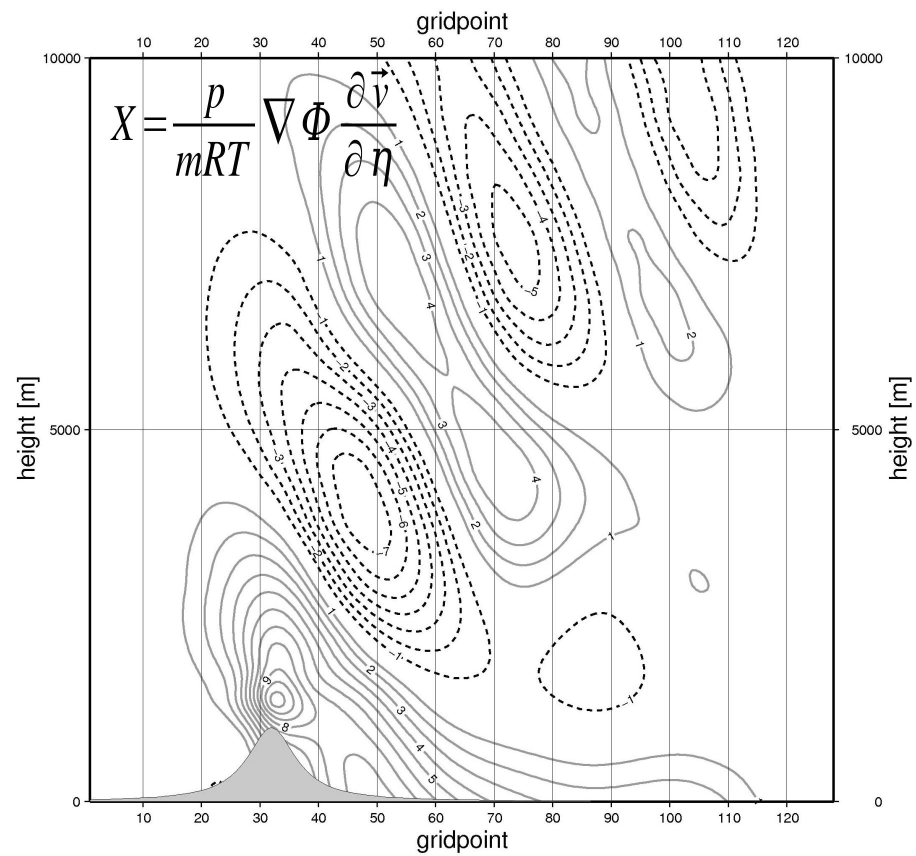
Ref VFE, 35lev, dt=5s
X-term FD, Z-term-VFE



min: -5.7843
max: 9.6362
step: 1

2006 Nov 30 16:36:42 experiment: VF18

Ref VFE, 35lev, dt=5s
X-term VFE, Z-term-FD



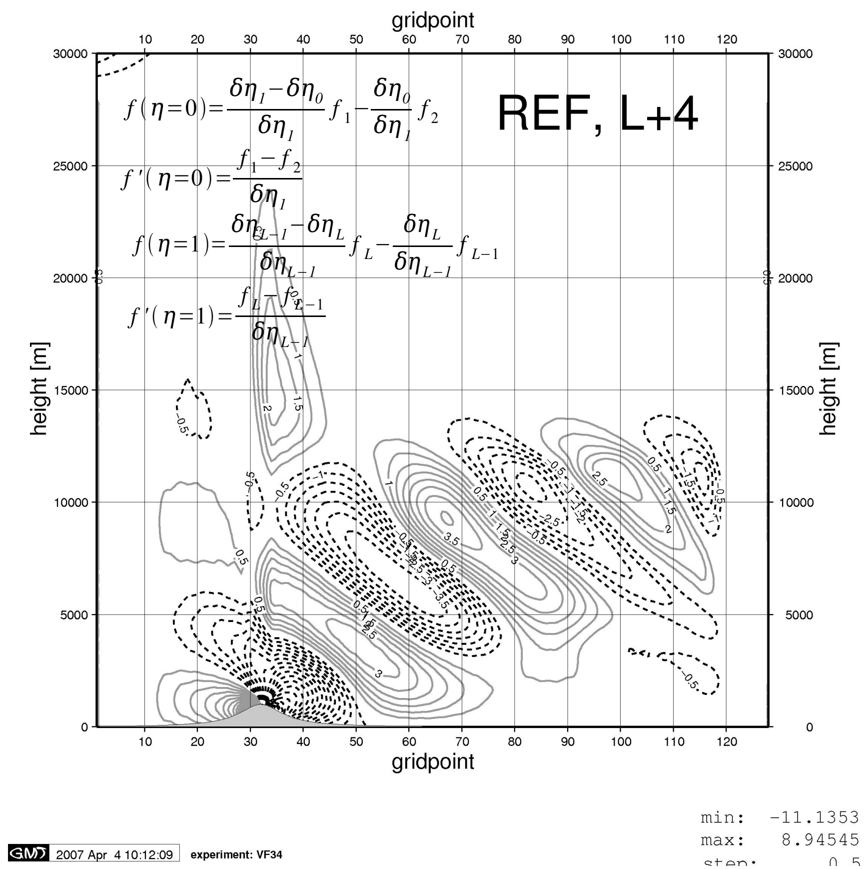
min: -7.6637
max: 11.46
step: 1

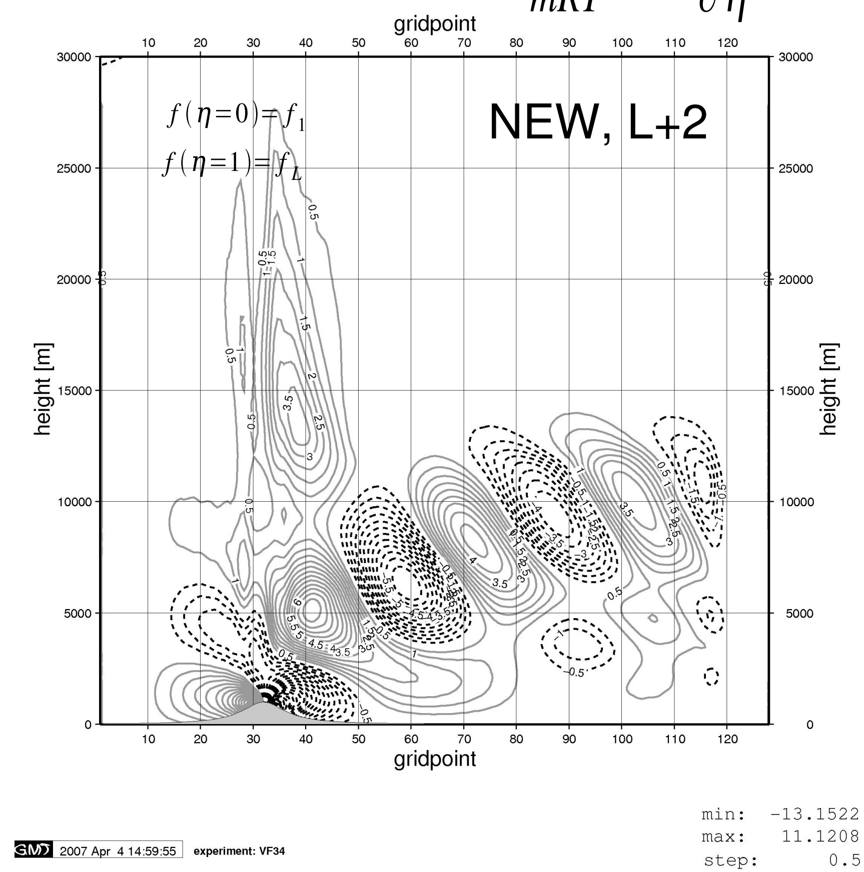
2006 Nov 30 20:30:34 experiment: VF19

VFE scheme – non-linear terms BC

Definition of extra derivative operator for non-linear term

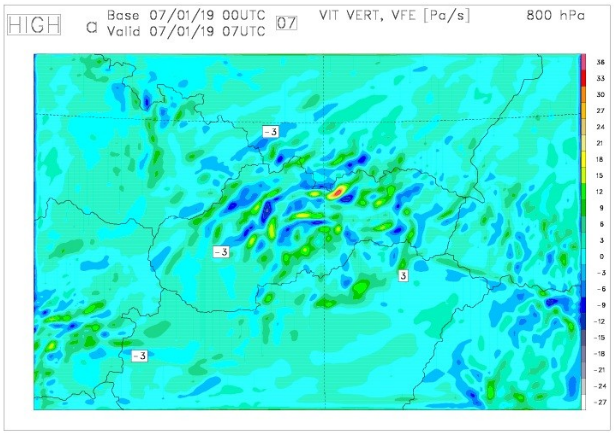
$$X = \frac{p}{mRT} \nabla \Phi \frac{\partial \vec{v}}{\partial \eta}$$





No choice of BC provided better results than REF

VFE results: 3D tests



- Model stable in first 3D test, dt=120s, res. 2.5 km, 37 vert. levels
- 19/01/07 0-12h, strong large scale flow over mountains
- 12h simulation

VFE simulation for 800 hPa height after 7h.

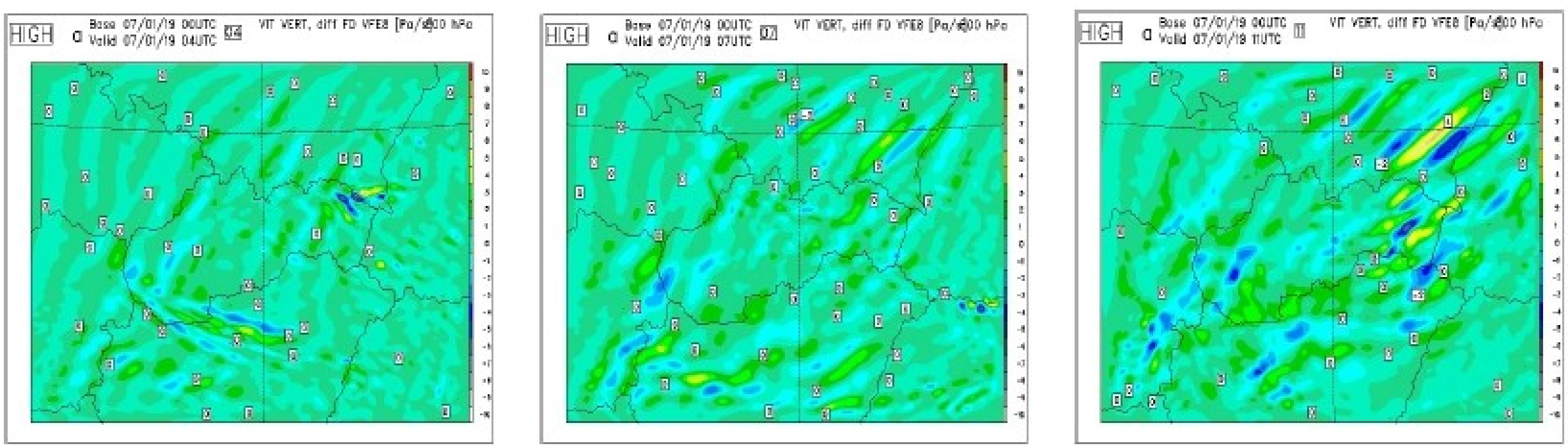
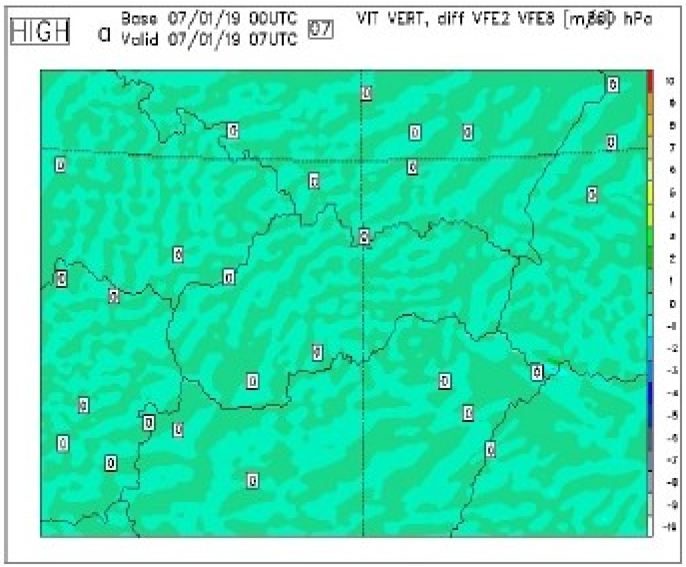


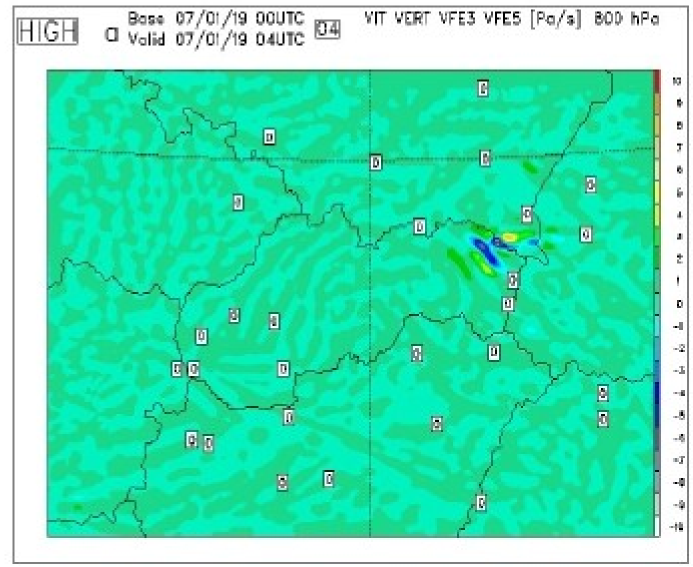
Figure 1: From left to right: Differences between FD and "complete" VFE simulations for a height of 800 hPa, after 04,07 and 11 hours.

VFE results: 3D tests

X-term VFE discretization influence



VFE (lapl+Zterm) – VFE (only integrals)



VFE (lapl+Xterm) – VFE (lapl)

VFE scheme – conclusions

- Analysis of stability in linear framework suggests that it is possible to extend HY model VFE scheme to NH model
- From stability point of view the crucial is the definition of vertical laplacian operator (all eigenvalues must be real and negative)
- Spectral solver is iterative one with one variable Helmholtz as a kernel, the convergence is very fast
- 3D tests are stable (dt=120s for dx=2.5km)
- The problem with BCs still remains and the noisy behaviour of system due to VFE discretization of Xterm must be solved