4. Martin GERA : "Improved representation of boundary layer"

Turbulence scheme

The topic is to find expressions for exchange coefficients which contain directly turbulent kinetic energy (TKE) handled as moderate prognostic variable. This feature allows us to dispose of exchange coefficients with a "history" of turbulence evolution. TKE equation employs virtual potential temperature. If we use potential temperature as thermodynamic variable, it allows us to degrade the number of parameterized equations. However the ALADIN model works with humidity fluxes. These fluxes can be expressed by conversion terms. Here one can discuss, if we do not lose a possibility to manipulate with (co)variances of potential temperature and specific humidity independently.

After introducing the parameterizations one can write a relation for fluxes :

$$\overline{u_i\theta} = \frac{L_k}{c_s\, e^{1/2}}\left[-\frac{2}{3}\, e\, \frac{\partial\overline{\theta}}{\partial x_i} + \frac{2}{3}\,\delta_{i3}\, g/\overline{\theta_v}\cdot\, \overline{\theta_v\theta}\right] \qquad (1)$$

$$\overline{u_i q} = \frac{L_k}{c_h\, e^{1/2}}\left[-\frac{2}{3}\, e\, \frac{\partial\overline{q}}{\partial x_i} + \frac{2}{3}\,\delta_{i3}\, g/\overline{\theta_v}\cdot\, \overline{\theta_v q}\right] \qquad (2)$$

$$\overline{\theta q} = \frac{L_\varepsilon}{2c_{\theta q}\, e^{1/2}}\left[-\overline{u_m\theta}\cdot\frac{\partial\overline{q}}{\partial x_m} - \overline{u_m q}\cdot\frac{\partial\overline{\theta}}{\partial x_m}\right] \qquad (3)$$

$$\overline{\theta^2} = \frac{L_\varepsilon}{c_\theta\, e^{1/2}}\left[-2\,\overline{u_m\theta}\cdot\frac{\partial\overline{\theta}}{\partial x_m}\right] \qquad (4)$$

$$\overline{q^2} = \frac{L_\varepsilon}{c_q\, e^{1/2}}\left[-2\,\overline{u_m q}\cdot\frac{\partial\overline{q}}{\partial x_m}\right] \qquad (5)$$

All notations are detailed at the end of the present paper. We can use next expression for the formulation of potential temperature fluxes :

$$\overline{u_i\theta_v} = E_\theta\, \overline{u_i\theta} + E_q\, \overline{u_i q} \qquad \text{with} \qquad E_\theta = \overline{\theta_v}/\overline{\theta} \qquad E_q = 0.61\overline{\theta} \qquad (6)$$

If we ask for the same form of potential temperature fluxes as introduced in formula (1), it seems straightforward to use the expression :

$$\overline{u_i\theta_v} = \frac{L_k}{c\, e^{1/2}}\left[-\frac{2}{3}\, e\, \frac{\partial\overline{\theta_v}}{\partial x_i} + \frac{2}{3}\,\delta_{i3}\, g/\overline{\theta_v}\cdot\, \overline{\theta_v\theta_v}\right] \qquad (7)$$

After manipulation with equations, we obtain a condition for coefficients c_s , c_h : $c = c_s = c_h$.

For the variance of potential temperature, we create a similar formula :

$$\overline{\theta_v{}^2} = \frac{L_\varepsilon}{c_0\, e^{1/2}}\left[-2\,\overline{u_m\theta_v}\cdot\frac{\partial\overline{\theta_v}}{\partial x_m}\right] \qquad (8)$$

In a similar manner, to retain a consistency between equations, we get a condition for coefficients c_θ, c_q and $c_{\theta q}$: $c_0 = c_\theta = c_q = 2c_{\theta q}$.

After solving the equations for fluxes we get an expression for exchange coefficients :

$$\overline{u_i\theta_v} = -K_h \frac{\partial\overline{\theta_v}}{\partial x_i} \tag{9}$$

where

$$K_H = \frac{2}{3}\frac{L_k}{c} e^{1/2}\,\Phi_i \qquad [m^2 \cdot s^{-1}] \tag{10}$$

The function of air stratification has form :

$$\Phi_i = 1 - \delta_{i3}\frac{2c_1 R'(\theta_v)^2}{R(\theta_v) + 2c_1 R(\theta_v)^2} \;\Rightarrow \tag{11}$$

$$\text{in 1D case : } R'(\theta_v) = R(\theta_v) \;\Rightarrow \Phi_3 = \frac{1}{1 + 2c_1 R(\theta_v)} \tag{12}$$

where : $c_1 = \frac{2}{3}\frac{1}{cc_0}$

The Redelsperger numbers R, R' depend on virtual potential temperature instead of potential temperature .

$$R(\theta_v) = g/\overline{\theta_v} \cdot \frac{L_\varepsilon L_k}{e} \cdot \frac{\partial\overline{\theta_v}}{\partial x_3}$$

$$R'(\theta_v) = g/\overline{\theta_v} \cdot \frac{L_\varepsilon L_k}{e} \cdot \left(\frac{\partial\overline{\theta_v}}{\partial x_m} \cdot \frac{\partial\overline{\theta_v}}{\partial x_m}\right)^{1/2}$$

Fluxes for potential temperature and humidity now read :

$$\overline{u_i\theta} = -K_H \frac{\partial\overline{\theta}}{\partial x_i} \tag{13}$$

$$\overline{u_i q} = -K_H \frac{\partial\overline{q}}{\partial x_i} \tag{14}$$

The form of expressions (7) and (8) predefine the setting of the coefficients. Redelsperger numbers contain humidity and potential temperature dependence concurrently. This form yields to the same exchange coefficients for humidity and potential temperature. Their fluxes (variances) only differ through variable gradient. Using equation (9) and :

$$\overline{\theta_v{}^2} = c_1\, L_\varepsilon L_k \left(\frac{\partial\overline{\theta_v}}{\partial x_m} \cdot \frac{\partial\overline{\theta_v}}{\partial x_m}\right)\Phi_m \tag{15}$$

together with TKE equation encloses our system of equations.

In a similar manner we express exchange coefficients for momentum fluxes :

$$\overline{u_i u_j} = \frac{2}{3}\,\delta_{ij}\,e \;-\; K_M\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\frac{\partial U_m}{\partial x_m}\right) \qquad (16)$$

$$K_M = \frac{4}{15}\frac{L_k}{c_M}\,e^{1/2} \qquad [m^2 \cdot s^{-1}] \qquad (17)$$

One may express now the local physical tendencies and vertical turbulent fluxes respecting the ALADIN formalism :

$$\left(\frac{\partial \mathrm{U}}{\partial t}\right)_{\psi} = -g\,\frac{\partial J_{\psi}}{\partial\, p} \qquad (18)$$

$$J_{\psi} = -K'_{\psi}\,\left|\Delta \mathrm{U}\right| \qquad [kg\cdot m^{-2}\cdot s^{-1}\cdot m\cdot s^{-1} \equiv N\cdot m^{-2}] \qquad (19)$$

$$K'_{\psi} = \overline{\rho}\; g\,\frac{K_{\psi}}{\Delta\Phi} \qquad [kg\cdot m^{-2}\cdot s^{-1}] \qquad (20)$$

In 1D situation, where only vertical exchanges are expected, the parameterized equation for TKE, (9) and (16) have the form (for more details see [2]) :

$$\frac{\partial e}{\partial t} = g/\overline{\theta_v}\;\cdot\overline{w\theta_v} - \overline{uw}\,\frac{\partial U}{\partial z} - \overline{vw}\,\frac{\partial V}{\partial z} - 1/\overline{\rho}\,\frac{\partial}{\partial z}\left(\overline{\rho}\;W\,e\right) - 1/\overline{\rho}\,\frac{\partial}{\partial z}\left(c_{2M}\,\overline{\rho}\,L_k\,e^{1/2}\,\frac{\partial e}{\partial z}\right) - c_{\varepsilon}\frac{e^{3/2}}{L_{\varepsilon}} \qquad (21)$$

$$\overline{w\theta_v} = -K_H\,\frac{\partial\overline{\theta_v}}{\partial\, z} \qquad (22)$$

$$\overline{uw} = -K_M\,\frac{\partial U}{\partial\, z} \qquad (23)$$

$$\overline{vw} = -K_M\,\frac{\partial V}{\partial\, z} \qquad (24)$$

Initialization

Doing an analysis of exchange coefficients, one could see a direct dependency on TKE. We said above that TKE will be a prognostic variable. It is apparent that we need for starting computations an initial value of TKE. We have no possibility without additional assumptions to compute subgrid values in the model. One possibility is to use a similar algorithm for the initialization as for the next computation. The initial value is computed from stationary simplification of the TKE equation, neglecting advection terms. One gets the following equation for initial TKE :

$$\frac{\partial e}{\partial t} = 0 = -(\overline{u_i u_j} - \frac{2}{3}\,\delta_{ij}e)\,\frac{\partial U_i}{\partial x_j} + g/\overline{\theta_v}\;\cdot\overline{u_3\theta_v} - \varepsilon \qquad (25)$$

Equation has this form after rearrangement :

$$e = \frac{L_{\varepsilon}}{c_{\varepsilon}}\left[\frac{4}{15}\,\frac{L_k}{c_M}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\frac{\partial U_m}{\partial x_m}\right)\frac{\partial U_i}{\partial x_j} + g/(\overline{\theta_v}\,e^{1/2})\cdot\overline{u_3\theta_v}\right] \qquad (26)$$

After solving a system of fluxes equations we get a value of TKE. From turbulent energy expression it is clear that eddy length parameters L_{ε}, L_k are unknown. We can see a lot of methods for determine these parameters in literature. We use a spectral characteristic for their determination. To study a wavenumber dependence on the spatial structure of TKE and air stratification function one has to use a relationship as :

$$E(k) = \frac{1}{2\pi}\int E^{*}(\omega)\ F(k-\omega)\ d\omega \tag{27}$$

where F is the Fourier transform of the air stratification function. From this expression one can see the influence of air stratification on the energy spectrum.

Near a wall, energy is blocked. From article [1] we can describe the deficit of turbulent kinetic energy, which is triggered by dissipation, using :

$$D_{\varepsilon} = \int_{k_M}^{\infty} E_F^{*}(k)\ dk \Big/ \int_{k_M}^{\infty} E_B^{*}(k)\ dk \tag{28}$$

D_{ε} measures the global deficit of TKE . E_F* is the spectral density of TKE in free atmosphere in neutrally stratified air (homogenous turbulence), E_B* is the spectral density of TKE nearest the surface (wall-bounded turbulence). These formulations allow us to compute length parameters and consecutively energy and fluxes. This kind of solution solves the problem while resolving vertical variance automatically.

Implementation to ALADIN

From the previous analysis it is apparent that the computed fluxes will depend on the distance to surface. Near the boundary, we have :

$$J_{\psi} = -K_{\psi}^{'}\ \Delta\psi \qquad\qquad K_{\psi}^{'} = \overline{\rho}\ C_{\psi}\ \left|\mathrm{U}(z)\right|$$

where C_{ψ} is an exchange coefficient at the surface. Formula (28) is applied on surface layer, which coincide with last interface (half level) and on the last but one half level. It is reason of definition L_{ε}, L_k nearest the surface. In the free atmosphere we use expression (18).

One can discuss now, that spectral characteristics are used for computation of some coefficients, although vertical grid is irregular. From linear spectral analysis it is apparent that the integral value of total TKE is unaltered with the change of density of the grid. The alteration is visible at intensity of $E(k)$ only. If we do the additional assumption that, in PBL, the density of mesh is quasi-regular, we can use for the derivation a mesh resolution nearest our point of computation. This method serves an estimation of the coefficient’s magnitude. Their value must be tuned.

Every next derivation is derived from the fact, that TKE is computed in half levels. The origin of computation is using the dry static energy or Richardson number expression. The moisture is included in the gas constant. It is a similar approach for the introduction of virtual temperature. Correction for shallow convection is well applicable in this moment. For Redelsperger number in 1D case, one has the following expression in ALADIN model :

$$R(\theta_v) = G(z)\ \frac{\Delta\phi\Delta S}{c_p T\ e} = G(z)(\Delta U_i)^2\ \frac{Ri}{e} \tag{29}$$

$$\text{where : }\ G(z) = \begin{cases} A_k(z)A_e(z)\dfrac{g[z+z_{0M}]\ g[z+z_{0H}]}{(\Delta\phi)^2} & z < 2L \\ 1 & z \geq 2L \end{cases} \tag{30}$$

G is a surface influence function, ΔS is a change of dry static energy and z_{0M}, z_{0H} are roughness length for momentum and thermal processes. For Ri we have expression :

$$Ri = \frac{\Delta\phi\Delta S}{c_p T\,(\Delta U_i)^2}$$

These knowledges are sufficient for the expression of our stability function Φ_i. For initialization of TKE we use analogous proceeding. One has a result :

$$e = \frac{4}{15 c_M \cdot c_\varepsilon} G(z)\,(\Delta U_i)^2 f(Ri) \qquad (31)$$

where $f(Ri)$ is stationary equivalent of Φ_i..

After taking into account our expressions for exchange coefficients (10,17), one can easy express these coefficients with present formulas. For computation of exchange coefficient we need to setup parameterization coefficients c, c_0 only. We suppose, that c_M, c_ε are set from spectral knowledge.

The direct dependency of the stratification function or Redelsperger number on Ri can cause some problems with value bounding . In region with strong instability or small wind shear it can produce difficulties. Limitation of Ri value is garbled from current ALADIN scheme for this reason.

PBL parameters at standard meteorological heights (METEO)

Kinematic turbulent fluxes could be expressed nearest the surface by friction, scaling velocity or by adequate thermal quantities. The computation assumes small change (10%) of fluxes in ground layer. Therefore we apply constant statistical moments at this case. The current ALADIN style of computation (application of logarithmic profiles and of stability function) require to know exchange coefficient at neutral stratification. In our case it is problem, because TKE contains directly stability feature and we can not do separation very easy. If we want to solve the problem realistically, we must solve a differential equation for TKE for a neutrally stratified atmosphere. This solution for getting TKE in a neutrally stratified atmosphere is expensive. Another approach, which exploits properties of stationary theory, can solve our problem satisfactory (it is implemented for beginning). For exchange coefficients in a neutrally stratified atmosphere one can now write the expressions :

$$K_H = \frac{2}{3}\frac{L_k}{c}\left(\frac{e}{f(Ri)}\right)^{1/2} \qquad (32)$$

$$K_M = \frac{2}{3}\frac{L_k}{c_M}\left(\frac{e}{f(Ri)}\right)^{1/2} \qquad (33)$$

In these formulas we do the same simplification for TKE like for friction fluxes. Energy will has a no vertical dependence at lowest levels nearest surface.

Notations

A bar denotes an averaging

δ_{ij}	... Kronecker symbol
index m	... Einstein’s summation ($X_m = X_1 + X_2 + X_3$) (see [1])
e	... turbulent kinetic energy
p	... pressure
q	... specific humidity
T	... temperature
θ	... potential temperature

θ_v	... virtual potential temperature
Φ	... geopotential
u_i	... perturbed wind components
U_i	... components of mean wind : U=(U_1, U_2, U_3)=(U, V, W)
S	... dry static energy
g	... gravitational acceleration
Ri	... Richardson number
c_p	... specific heat at constant pressure
L_ε, , L_k	... dissipitation and eddy viscosity length scales (see [1])
c_s , c_h , *etc..*	... dimensionless parameterization coefficients
K_H , K_M	... exchange coefficients for heat and momentum
z_{0M}, z_{0H}	... roughness lengths for momentum and thermal processes
Φ_i	... air stratification function
$f(Ri)$	... stationary equivalent of Φ_i (for more details see [2])
R	... Redelsperger number (as a function of virtual potential temperature)
R'	... tridimensional Redelsperger number
E	... spectral dependency of TKE
E^*	... Fourier transform of TKE in neutrally stratified air
F	... Fourier transform of TKE in neutrally stratified air
ε	... viscous dissipation of TKE

Bibliography

[1] J.L. Redelsperger, F. Mahe, P. Carlotti, 2000 : A simple and general subgrid model suitable both for surface layer and free-stream turbulence, *Flow, Turbulence, and Combustion*, Vol. 200, No.66, pp.453-475.

[2] J. Cuxart, P. Bougeault, J.L. Redelsperger, 2000 : A turbulence scheme allowing for mesoscale and large-eddy simulations, *Quarterly Journal of the Royal Meteorological Society*, Vol. 126, No.562, pp.1-30.